NEVANLINNA'S FIVE VALUE THEOREM

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ABSTRACT

If f and g are meromorphic functions of one complex variable and f and g have the same inverse images(ignoring multiplicities) on five distinct values then they are identically equal, i.e $f \equiv g$. It is well known that any polynomial is determined by its zero points except for a non-constant factor, but this is not true for transcendental entire or meromorphic functions. For example, we have the functions e^z and e^{-z} have the same value $-1, 1, 0$ and ∞ points. This 1929 result of Rolf Nevanlinna $(1895 - 1980)$ was described by the distinguished analyst Lee Rubel as his favourite in all mathematics. It follows from the second fundamental theorem of Nevanlinna Theory, in turn described by the Hermann Weyl as one of the greatest achievements of twentieth century mathematics.

We will discuss the Jensen's theorem and then talk about the Nevanlinna's First Fundamental Theorem and Second Fundamental Theorem. We'll see some examples and remarks regarding the theorems we proof, some examples to strengthen our understanding on this topic and then finally discuss about the Nevanlinna's Five Value theorem.

1 Introduction

We will use Jensen's theorem to explain the First Fundamental theorem of Nevanlinna's theory. We will define some terms to help us explain it.We will prove the Jensen's theorem using the Gauss Mean Value Theorem.

Theorem 1.1. *(*Gauss Mean Value Theorem) Suppose u is a harmonic function in $\mathbb{D}$. Then the value of u at the center is equal to the average of the boundary values of u. That is,

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})$$

Proof. Suppose we have an analytic function $f(z)$ with real part $u(x, y)$. We can then apply Cauchy's Integral Formula to evaluate the value of f at zero and then take the real part of the integral. ∎

Theorem 1.2. *(*Jensen's Theorem*) If f is meromorphic in $|z| \leq R$, if $r < R$, and if*

$$f(z) = a_k z^k + a_{k+1} z^{k+1} + \cdots (a_k \neq 0)$$

is the Laurent expansion of f around zero, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta = \log |a_k| + \sum_{r_n \leq r} \log \frac{r}{r_n} - \sum_{\rho_n \leq r} \log \frac{r}{\rho_n} + k \log r,$$

where the zeros of f are $z_j = r_j e^{i\theta_j}$ and the poles of f are $w_i = p_j e^{i\phi_j}$, not counting zeros or poles at the origin.

Proof. Without loss of generality let us assume $f(0) = 1$ and $R = 1$. Then

$$F(z) = \frac{f(z)}{\prod_n \left(\frac{z - z_n}{1 - \overline{z}_n z} \right)} \cdot \prod_n \frac{z - w_n}{1 - \overline{w_n} z}$$

We know that F is an analytic function with no zeroes and no poles. Since $\log|F|$ is harmonic function in the disk we can use the Gauss Mean Value theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|F(re^{i\theta})|d\theta = \log|f(0)| + \sum\log\rho_n - \sum\log r_n.$$

But we have,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|F(re^{i\theta})|d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|d\theta.$$

because we have

$$\left|\frac{z-z_0}{1-\overline{z_0}z}\right| \text{ equal to 1 on } |z|=1 \text{ if } |z_0|<1.$$

Let $n(r,f)$ denote the number of poles of f in the closed disk $|z|\leq r$, counted according to multiplicity. Thus, we say that $\left(r,\frac{1}{f-a}\right)$ count the number of a-points. Now, let us take $k^+=\max(k,0)$ and $k^-=\min(k,0)$, so that $k^+-k^-=k$. Then, we have

$$\sum_{r_n\leq r}\log\frac{r}{r_n}+k^+\log r = \int_{0^+}^{r}\log\frac{r}{t}d[(n(t)-n(0)]+k^+\log r$$

Substituting $u=\log\frac{r}{t}$ and $v=n(t)-n(0)$ then $du=-\frac{dt}{t}$. Now, we can apply integration by parts to solve further

$$[n(t)-n(0)]\log\frac{r}{t}\bigg|_{0^+}^{r}+\int_{0^+}^{r}\frac{n(t)-n(0)}{t}dt+k^+\log r$$

Let us define

$$N\left(r,\frac{1}{f}\right)\equiv k^+\log r+\int_{0^+}^{r}\frac{n\left(t,\frac{1}{f}\right)-n\left(0,\frac{1}{f}\right)}{t}dt=\sum_{r_n\leq r}\log\frac{r}{r_n}+k^+\log$$
$$N(r,f)\equiv k^-\log r+\int_{0^+}^{r}\frac{n(t,f)-n(0,t}{t}dt=\sum_{\rho_n\leq r}\log\frac{r}{\rho_n}+k^-\log r.r$$

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$n(r,f)$ counts the number of poles f in the disk $|z|\leq r$. We usually normalize f so that we have the value $f(0)=1$, and for $k^+=k^-=n(0,f)=n\left(0,\frac{1}{f}\right)=\log|a_k|=0$.

2 First Fundamental theorem of Nevanlinna's theory

Rewriting Jensen's theorem, we get:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|d\theta=\log|a_k|+N\left(r,\frac{1}{f}\right)-N(r,f)$$

where N is a kind of average number of poles of f.

As we know that

$$\log x=\log^+ x-\log^+\frac{1}{x}$$

So we may write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta-\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+\left|\frac{1}{f(re^{i\theta})}\right|.$$

Let $m=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta$. Rewriting the Jensen's theorem as following, we will have

$$m(r,f)-m\left(r,\frac{1}{f}\right)=\log|a_k|+N\left(r,\frac{1}{f}\right)-N(r,f)$$

Remark 2.1. The notation $N(r, f)$ counts the number of poles of f(with some certain kind of averaging). In other words it is the averaged number of times f takes the value ∞. The notation $m(r, f)$ measures the tendancy of f to take the value ∞. There is a slight difference in both the terms. In some cases, the quantity $m(r, f) + N(r, f)$ measures the tendancy of f to take the value ∞. Similarly, the quantity $m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right)$ will measure the tendancy of f to take the value 0. So the above version of Jensen's Theorem asserts that the total affinity of f for ∞ is the same as the total affinity of f for the value zero, modulo a bounded function of r.

First fundamental theorem is basically based on the observation that for any constant value of a, the affinity of $f - a$ is ∞ is essentially the same as that for f when the affinity of f is ∞ and the the affinity $f - a$ is zero when the affinity of f is a.

Fix $a \in \mathbb{C}$. Then we have $N(r, f) = N(r, f - a)$ since z is the pole of f if and only if it is the pole of $f - a$. Then using the property

$$|\log^+ |x - a| - \log^+ |x|| \leq \log^+ |a| + \log 2$$

we will have

$$|m(r, f - a) - m(r, f)| \leq \log^+ |a| + \log 2.$$

We define another notation that is

$$T(r, f) = N(r, f) + m(r, f)$$

where is T is known as the Nevanlinna characteristic of f.

Lemma 2.2. *Let $f(z)$ be a meromorphic function in $|z| \leq R$ (where R can be from 0 to ∞) with the following Laurent expansion in the neighbourhood of the origin,*

$$f(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \cdots, c_\lambda \neq 0.$$

Then for $0 < r < R$, we have

$$T(r, f) = T\left(r, \frac{1}{f}\right) + \log |c_\lambda|$$

Remark 2.3. It is known as Jensen-Nevanlinna formula, which is another expression of Jensen's formula and exhibits relation between characteristic functions $f(z)$ and $1/f(z)$.

Proof. The first fundamental theorem is a rephrasing of the Jensen's theorem using the notations that we introduced.

Usually to work with modulo bounded functions of r, we may sometimes abuse the notation and write wrong notations. The characteristic plays a central role in the theory of meromorphic (and entire) functions. ∎

3 The Second Fundamental Theorem

We will use a lemma which we'll use to prove the Second Fundamental Theorem.

Lemma 3.1. *Suppose that $f(z)$ is a non-constant meromorphic function in $|z| < R$ and $a_j (j = 1, 2, 3, 4, 5)$ are q distinct finite complex numbers. Then*

$$m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) = \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + O(1)$$

holds for $0 < r < R$.

Proof. Let us take

$$F(z) = \sum_{j=1}^{q} \frac{1}{f(z) - a_j}$$

Now using the relation

$$m\left(r, \sum_{j=1}^{p} f_i\right) \leq \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + \log q$$

Then,

$$m(r,F) \leq \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) + \log q$$

Now we will search for a lower bound for $m(r,F)$. For finding a lower bound for $m(r,F)$ we let

$$\min_{1\leq j\leq k\leq q} |a_j - a_k| = \delta.$$

Clearly we can say that $\delta > 0$. We can then make 2 cases for a fixed point z.

In the first case let us suppose that there exists some integer $k(1 \leq k \leq q)$ in $j = 1,2,3,\cdots,q$ such that we have

$$|f(z) - a_k| < \frac{\delta}{2q}.$$

For $j \neq k$, using Triangle inequality we will get

$$\begin{aligned} |f(z) - a_j| &= |f(z) - a_k + a_k - a_j| \\ &\geq |a_k - a_j| - |f(z) - a_k| \\ &\geq \delta - \frac{\delta}{2q} = \frac{2q-1}{2q}\delta \end{aligned}$$

Using this result and our initial condition for proving first case we have

$$\frac{1}{|f(z)-a_j|} \leq \frac{2q}{(2q-1)\delta} < \frac{1}{2q-1} \cdot \frac{1}{f(z)-a_k}$$

Hence, we will get

$$\begin{aligned} |F(z)| &\geq \frac{1}{|f(z)-a_k|} - \sum_{j=1(j\neq k)}^{q} \frac{1}{|f(z)-a_j|} \\ &\geq \frac{1}{|f(z)-a_k|} - \frac{q-1}{2q-1} \cdot \frac{1}{|f(z)-a_k|} \\ &> \frac{1}{2|f(z)-a_k|} \end{aligned}$$

which yields

$$\log^+ |F(z)| > \log^+ \frac{1}{f(z)-a_k} - \log 2.$$

This result implies

$$\sum_{j=1(j\neq k)}^{q} \frac{1}{|f(z)-a_j|} \leq \sum_{j=1(j\neq k)}^{q} \frac{2q}{(2q-1)\delta} < q \log^+ \frac{2q}{\delta}.$$

This result follows from

$$\log^+ |F(z)| > \sum_{j=1}^{q} \log^+ \frac{1}{|f(z)-a_j|} - q\log^+ \frac{2q}{\delta} - \log 2. \tag{3.1}$$

Now we'll prove the second case supposing that we have the inequality holds $j = 1,2,3,\cdots,q$.

$$|f(z) - a_j| \leq q \log^+ \frac{2q}{\delta}.$$

Hence, we can see that our relation still holds true in (3.1). Therefore, the result is also true for the second case.

Replacing z by $re^{i\theta}$ in the equation (3.1) and then integrating both sides with respect to θ from the limits 0 to 2π we will get

$$m(r,F) \geq \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) - q\log^+ \frac{2q}{\delta} - \log 2.$$

Combining this result with our previous result

$$m(r,F) \leq \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) + \log q$$

gives us our result. This completes our proof for the lemma. ∎

Now that we've proved this lemma we are now finally ready to prove the Second Fundamental Theorem.

Theorem 3.2. *(Second Fundamental Theorem) Suppose that $f(z)$ is a non-constant meromorphic function in $|z| < R$ and $a_j(1,2,3,4,5)$ are $q(\geq 2)$ distinct finite complex numbers. Then for $0 < r < R$, we have*

$$m(r,f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) \leq 2T(r,f) - N_1(r,f) + S(r,f),$$

where

$$N_1(r,f) = 2N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right)$$

and

$$S(r,f) = m\left(r, \frac{f'}{f}\right) + m\left(r, \sum_{j=1}^{q} \frac{f'}{f-a_j}\right) + O(1).$$

Proof. Let

$$F(z) = \sum_{j=1}^{q} \frac{1}{f-a_j}$$

In terms of Lemma 3.1 we have the relation

$$m(r,F) = \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) + O(1) \tag{3.2}$$

Using the other relation we get

$$\begin{aligned} m(r,F) &\leq m(r,f'F) + m\left(r, \frac{1}{f'}\right) \\ &\leq m(r,f'F) + T(r,f') - N\left(r, \frac{1}{f'}\right) + O(1). \end{aligned} \tag{3.3}$$

Since,

$$\begin{aligned} T(r,f') &= m(r,f') + N(r,f') \\ &\leq m(r,f) + m\left(r, \frac{f'}{f}\right) + N(r,f') \\ &= T(r,f) + +m\left(r, \frac{f'}{f}\right) + N(r,f') - N(r,f), \end{aligned} \tag{3.4}$$

Therefore, our following result follows from (3.2), (3.3), (3.4).

$$\begin{aligned} m(r,f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) \leq 2T(r,f) - &\left\{2N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right)\right\} + \\ &m\left(r, \frac{f'}{f}\right) + m\left(r, \sum_{j=1} \frac{f'}{f-a_j)}\right) + O(1) \end{aligned}$$

∎

which completes our proof for Second Fundamental Theorem.

4 Nevanlinna's Five Value Theorem

We are now finally ready to introduce the Nevanlinna's Five Value Theorem. This result is one of the most important results of Nevanlinna on the uniqueness of meromorphic functions. After Nevanlinna's five value theorem, there are vast references on the uniqueness of meromorphic functions sharing values and sets in the whole complex plane. It is an interesting topic how to extend some important uniqueness results in the complex plane to an angular domain or the unit disc.

Theorem 4.1. *(*Nevannlina's Five Value Theorem) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and $a_j(j = 1, 2, 3, 4, 5)$ be five distinct values in the extended complex plane. If f and g share $a_j(j = 1, 2, 3, 4, 5)$ ignoring multiplicties, then $f \equiv g$.

Proof. We assume that $a_j(j = 1, 2, 3, 4, 5)$ are all finite. Using second fundamental Theorem we get

$$3T(r, f) < \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f)$$
$$3T(r, g) < \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g - a_j}\right) + S(r, g)$$

Let us assume to the contrary that $f \not\equiv g$. Then by our assumption that f and g share $a_j(j = 1, 2, 3, 4, 5)$ ignoring multiplicities, we will get

$$\begin{aligned}\sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f - a_j}\right) = \sum_{j=1} \overline{N}\left(r, \frac{1}{g - a_j}\right) &\leq N\left(r, \frac{1}{f - g}\right)\\ &\leq T(r, f - g) + O(1)\\ &\leq T(r, f) + T(r, g) + O(1)\end{aligned}$$

Using our result that we used to prove our first assumption from the Second Fundamental Theorem we have

$$3(T(r, f) + T(r, g)) \leq 2(T(r, f) + T(r, g)) + S(r, f) + S(r, g).$$

Rewriting we get

$$T(r, f) + T(r, g) \leq S(r, f) + S(r, g).$$

This a contradiction. Hence, we proved the result $f \equiv g$.

Now let us assume that one of the values from $a_j(j = 1, 2, 3, 4, 5)$ is infinity. Without loss of generality let us take the value of $a_5 = \infty$. We can then take any finite value from $a_j(j = 1, 2, 3, 4)$ and set

$$F(z) = \frac{1}{f(z) - a}, \quad G(z) = \frac{1}{g(z) - a}$$

Then we take

$$b_j = \frac{1}{a - a_j} \quad (j = 1, 2, 3, 4) \text{ where } b_5 = 0.$$

Hence, we can conclude that F and G share $b_j(j = 1, 2, 3, 4, 5)$ ignoring multiplicities. Therefore, we can say that $F(z) \equiv G(z)$ and finally $f(z) \equiv g(z)$. ∎

Remark 4.2. We cannot weaken the our result for f and g sharing four values. For example, $f(z) = e^z$ and $g(z) = e^{-z}$ share four values $0, 1, -1, \infty$ ignoring multiplicities but $f(z) \not\equiv g(z)$.

We can now talk more about Nevanlinna's Five Value theorem. Li and Qiao proved a small function version of Nevanlinna's five-value theorem, which says that if two meromorphic functions share five small functions, then these two functions are identical which is the same as our definition of Nevanlinna's Five Value Theorem. Recently Theorem 4.1 was improved by C.C. Yang when ey proved the following theorem.

Let us denote $h(z)$ as a non-constant meromorphic function and a as an arbitrary complex number. We denote the zero set of $h(z) - a$ with the notation $\overline{E}(a, h)$, where each zero is counted only once.

Theorem 4.3. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $a_j (j = 1, 2, 3, 4, 5)$ be five distinct values. If*

$$\overline{E}(a_i, f) \subseteq \overline{E}(a_i, g) \qquad (j = 1, 2, 3, 4, 5)$$

and

$$\lim_{r\to\infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right) > \frac{1}{2},$$

then $f(z) \equiv g(z)$.

Proof. We assume without the loss of generality that all the values of $a_j (j = 1, 2, 3, 4, 5)$. Using Second Fundamental Theorem we can say that

$$3T(r, f) < \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r, f)$$

$$3T(r, g) < \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right) + S(r, g)$$

Now, assume to the contrary that $f(z) \not\equiv g(z)$. Then we have

$$\sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \le \overline{N}\left(r, \frac{1}{f-g}\right) \le T(r, f) + T(r, g) + O(1).$$

Again substituting the result from the previous line that we got from the Second fundamental theorem we have

$$\sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \le \left(\frac{1}{3} + o(1)\right) \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) + \left(\frac{1}{3} + o(1)\right) \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right)$$

Therefore, we can rewrite it as

$$\left(\frac{2}{3} + o(1)\right) \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \le \left(\frac{1}{3} + o(1)\right) \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right)$$

Therefore, we finally have,

$$\lim_{r\to\infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right) \le \frac{1}{2},$$

This contradicts with our initial assumption. Hence, we have the result $f(z) \equiv g(z)$. ∎

There is also a four-value theorem of Nevanlinna. If two meromorphic functions, $f(z)$ and $g(z)$, share four values counting multiplicities, then $f(z)$ is a Möbius transformation of $g(z)$. We simply say “2 CM+2 IM implies 4 CM”. So far it is still not known whether “1 CM +3 IM implies 4 CM", where the meaning of CM is counting multiplicities and IM means ignoring multiplicities.

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