ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Elliptic partial differential equations are a specific group of partial differential equations which have applications in various fields of physics. This paper will explore properties of specific elliptic partial differential equations such as the Laplace equation.

1. Introduction to Partial Differential Equations

Partial differential equations (commonly abbreviated as PDE) are equations that show the relationship between some unknown function with several variables and its partial derivatives. Let $k \geq 1$ be some integer and let U be an open subset of $\mathbb{R}^n$. Then, we say that PDEs are equations of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0 \qquad (x \in U) \tag{1}$$

where $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$ is given, $u : U \to \mathbb{R}$ is the unknown function, and $D^k u(x)$ is the set of all partial derivatives of order k. This equation is called a k^{th}-order PDE. In order to solve this PDE, we must find all u such that equation 1 is satisfied and the goal is to obtain simple, explicit solutions or to prove that such solutions exist and their properties are known. Sometimes, there are also additional conditions which must be met. In addition, a PDE can be classified as linear, semilinear, quasilinear, or fully nonlinear depending on its form for given functions $a_\alpha(|\alpha| \leq k)$ and f as follows:

Definition 1.1. The PDE 1 is *linear* if it has the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x).$$

If $f \equiv 0$, then this linear PDE is *homogeneous*.

Definition 1.2. The PDE 1 is *semilinear* if it has the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

Definition 1.3. The PDE 1 is *quasilinear* if it has the form

$$\sum_{|\alpha| \leq k} a_\alpha(D^{k-1}u, \ldots, Du, u, x) + a_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

Definition 1.4. The PDE 1 is called *fully nonlinear* if it depends nonlinearly upon the highest order derivatives.

A system of PDEs is defined as follows:

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Definition 1.5. Given $F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m$, an equation of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0 \qquad (x \in U) \tag{2}$$

is a k^{th}-order system of partial differential equations where

$$u : U \to \mathbb{R}^m, u = (u^1, \ldots, u^m)$$

is unknown.

Usually, the system comprises the same number m of scalar equations as unknowns $(u^1, \ldots, u^m)$ which is shown in our definition. However, it is possible for a system to have a different number of equations compared to unknowns[Olv14]. Lastly, when we explore the Laplace equation, a specific PDE, we will need harmonic functions as defined below.

Definition 1.6. A function $u : U \to \mathbb{R}$ is *harmonic* on U if $u \in C^2(U)$ and $\Delta u = 0$ in U. A function whose first and second derivatives both exist and are continuous is said to be of class C^2. The set of all harmonic functions in U is denoted as $H(U)$.

Harmonic functions hold many properties related to the Laplace equation including the mean-value formulas and its consequential results: the maximum principle, and uniqueness. We will only prove these three but there are many others such as smoothness.

2. Elliptic Partial Differential Equations

We will specifically be looking at elliptic PDEs which are a type of second-order PDE. Second-order PDEs are widely used in mechanics with parabolic and hyperbolic PDEs in addition to elliptic PDEs which are defined as follows.

Definition 2.1. Suppose we have a second-order PDE of the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y)$$

where x and y are independent variables. Then, this PDE is *elliptic* if $b^2 - 4ac < 0$, *hyperbolic* if $b^2 - 4ac > 0$, and *parabolic* if $b^2 - 4ac = 0$.

An important area of study with elliptic PDEs is the boundary value problem [Med18]

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{3}$$

where U is an open, bounded subset of $\mathbb{R}^n$ and $U : \overline{U} \to \mathbb{R}$ is the unknown $u = u(x)$. Here $f : U \to \mathbb{R}$ is given, and L is a second-order partial differential operator with one of the following forms:

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum i = 1^n b^i(x)u_{x_i} + c(x)u \tag{4}$$

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum i = 1^n b^i(x)u_{x_i} + c(x)u \tag{5}$$

for given coefficient functions a^{ij}, b^i, and c for $i, j = 1, \ldots, n$. Here, we write u_{x_i} for $\frac{\partial u}{\partial x_i}$ and similarly, $u_{x_i x_i}$ for $\frac{\partial^2 u}{\partial x_i^2}$.

Definition 2.2. We say that the PDE $Lu = f$ is in *divergence form* if L is given by 4 and in *nondivergence form* when L is given by 5.

There are three theorems that outline the existence of weak solutions of the boundary value problem.

Theorem 2.1. (First Existence theorem for weak solutions) *There is a number $\gamma \geq 0$ such that for each*

$$\mu \geq \gamma$$

and each function

$$f \in L^2(U),$$

there exists a unique weak solution $u \in H_0^1(U)$ of the boundary value problem.

Examples of elliptic PDEs include the Poisson equation and its special case, the Laplace equation which we will explore in the next section. For now, let's look at the properties of second-order elliptic PDEs. One of the most important theorems is the maximum principle which we will first present a weak form of [PR05].

Theorem 2.2. The weak maximum principle *Let D be a bounded domain, and let $u(x, y) \in C^2(D) \cap C(\overline{D})$ be a harmonic function in D. Then the maximum of u in $\overline{D}$ is achieved on the boundary ∂D.*

Proof. Consider a function $v(x, y) \in C^2(D) \cap C(\overline{D})$ satisfying $\Delta v > 0$ in D. We claim that v cannot have a local maximum point in D. To see why, recall from calculus that if $(x_0, y_0) \in D$ is a local maximum point of v, then $\Delta v \leq 0$, which contradicts our assumption. Since u is harmonic, the function $v(x, y) = u(x, y) + \varepsilon(x^2 + y^2)$ satisfies $v > 0$ for any $\varepsilon > 0$. Set $M = \max_{\partial D} u$, and $L = \max_{\partial D}(x^2 + y^2)$. From our argument about v it follows that $v \leq M + \varepsilon L$ in D. Since $u = v - \varepsilon(x^2 + y^2)$, it now follows that $u \leq M + \varepsilon L$ in D. Because ε can be made arbitrarily small, we obtain $u \leq M$ in D. □

The weak maximum principle does not exclude the possibility of the maximum (or minimum) of u being attained at an internal point. The stronger version that we will soon prove takes care of this by asserting that if u is not constant, then the maximum (and minimum) cannot be obtained at an interior point. To prove this, we need an important property of harmonic functions [PR05].

Theorem 2.3. (The mean value principle) *Let D be a planar domain, let u be a harmonic function there, and (x_0, y_0) be a point int D. Suppose B_R is a disk of radius R centered at (x_0, y_0), fully contained in D. For any $r > 0$, set $C_r = \partial B_r$. Then the value of u at (x_0, y_0) is the average of the values of u on the circle C_R:*

$$\begin{aligned} u(x_0, y_0) &= \frac{2\pi R}{\oint}_{C_R} u(x(s), y(s)) ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta) d\theta. \end{aligned}$$

Proof. Let $0 \leq r \leq R$ and define $v(r, \theta) = u(x_0 + r\cos\theta, y_0 + r\sin\theta)$. We also define the integral of v with respect to θ:

$$V(r) = \frac{1}{2\pi r}\oint_{C_r} v ds = \frac{1}{2\pi}\int_0^{2\pi} v(r,\theta)d\theta.$$

Differentiating with respect to r, we have

$$\begin{aligned} V_r(r) &= \frac{1}{2\pi}\int_0^{2\pi} v_r(r,\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial r}u(x_0 + r\cos\theta, y_0 + r\sin\theta)d\theta \\ &= \frac{1}{2\pi r}\oint_{C_r} \partial_n u ds = 0, \end{aligned}$$

where the last equality is from the following property of harmonic functions (solutions of the Laplace equation)

$$\int_\Gamma \partial_n u ds = 0$$

for any closed curve Γ fully contained in D. We will not prove this now. Therefore $V(r)$ does not depend on r, and thus

$$u(x_0, y_0) = V(0) = \lim_{p\to 0} V(p) = V(r) = \frac{2\pi R}{\oint}_{C_R} u(x(s), y(s))ds$$

for all $0 \leq r \leq R$ □

We also note that the converse is true which we can prove by contradiction.

Theorem 2.4. *Let u be a function in $C^2(D)$ satisfying the mean value property at every point in D. Then u is harmonic in D.*

Proof. Assume by contradiction that there is a point (x_0, y_0) in D where $\Delta u(x_0, y_0) \neq 0$. Without loss of generatily assum $\Delta u(x_0, y_0) > 0$. Since $\Delta u(x_0, y_0)$ is a continuous function, then for a sufficiently small $R > 0$, there exists in D a disk B_R of radius R, centered at (x_0, y_0) such that $\Delta u > 0$ at each point in B_R. Denote the boundary of this disk by C_R. It follows that

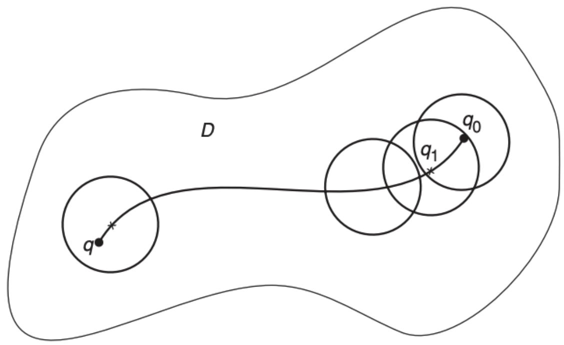
$$\begin{aligned} 0 < \frac{1}{2\pi}\int_{B_R} \Delta u dxdy &= \frac{1}{2\pi}\oint_{C_R} \partial_n u ds \\ &= \frac{R}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial R}u(x_0 + R\cos\theta, y_0 + R\sin\theta)d\theta \\ &= \frac{R}{2\pi}\frac{\partial}{\partial R}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)d\theta \\ &= R\frac{\partial}{\partial R}[u(x_0, y_0)] = 0. \end{aligned}$$

The last inequality comes from the assumption that u satisfies the mean value property and thus we have a contradiction. □

With the mean value property, we can prove the strong version of the maximum principle [PR05].

Theorem 2.5. (The strong maximum principle) *Let u be a harmonic function in a domain D (here we also allow for unbounded D). If u attains it maximum (minimum) at an interior point of D, then u is constant.*

Figure 1. A construction for the proof of the strong maximum principle.



Proof. Assume by contradiction that u obtains its maximum at some interior point q_0. Let $q \neq q_0$ be an arbitrary point in D. Let l be a smooth orbit in D connecting q_0 and q as shown in Figure 1. In addition, let d_l be the distance between l and ∂D.

Consider a disk B_0 of radius $d_{l/2}$ around q_0. From the definition of d_l and from the mean value theorem, we infer that u is constant in B_0 (since the average of a set cannot be greater than all the objects of the set). Select now a point q_1 in $l \cap B_0$, and denote by B_1 the disk of radius $d_{l/2}$ centered at q_1. From our construction it follows that u also reaches its maximal value at q_1. Thus we obtain that u is constant also in B_1. We continue in this way until we reach a disk that includes the point q. We conclude $u(q) = u(q_0)$, and since q is arbitrary, it follows that u is constant in D. Notice that we may choose the points $q_0, q_1, \dots,$ such that the process involves a finite number of disks $B0, B1, \dots, B_{n_l}$ because the length of l is finite, and because all the disks have equal radius. □

3. The Laplace Equation

The Laplace equation is a linear elliptic PDE that is a special case of the Poisson equation. Laplace's equation is written as

$$\Delta u = 0$$

where Δ is the Laplace operator (a special case of the elliptic operator) which is defined as the sum of all the second partial derivatives of u

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

Here, we can also write u_{x_i} for $\frac{\partial u}{\partial x_i}$ and similarly, $u_{x_i x_i}$ for $\frac{\partial^2 u}{\partial x_i^2}$. Poisson's equation is

$$\Delta u = f$$

where f is given and u is solved for but we will only be exploring Laplace's equation. One property of the Laplacian is the it is invariant under rotation. This means that if $\Delta u = 0$ and $v = U(Rx)$ where Rx is a rotation of x, then $\Delta v = 0$ as well. Therefore, we should find solutions that are also invariant under rotation, namely radial functions. This idea brings us the fundamental solution of the Laplace equation which are of the form

$$u(x) = v(|x|)$$

where $v : \mathbb{R} \to \mathbb{R}$ is to be found and only depends on the absolute value of x since it is radial.

3.1. **The Fundamental Solution.** When solving partial differential equations, it is helpful to first find explicit solutions, especially ones with special properties such as symmetry, and then expand them to more complicated solutions. The following method [Eva10] finds a solution u of the Laplace equation in $U = \mathbb{R}^n$ of the form

$$u(x) = v(r)$$

where $r = |x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ and v should be chosen to satisfy $\Delta u = 0$. Note that for $i = 1, \ldots, n$:

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(x_1^2 + \ldots + x_n^2\right)^{-1/2} 2x_i = \frac{x_i}{r} \qquad (x \neq 0).$$

Therefore,

$$u_{x_i} = v'(r)\frac{x_i}{r}, u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

for $i = 1, \ldots, n$. We thus have

$$\Delta u = v''(r) + \frac{n-1}{r}v'(r)$$

so $\Delta = 0$ if and only if

$$v'' + \frac{n-1}{r}v' = 0.$$

If $v' \neq 0$, we have

$$\log(v')' = \frac{v''}{v'} = \frac{1-n}{r}$$

so $v'(r) = \frac{a}{r^{n-1}}$ for some constant a. Then if $r > 0$, we have

$$v(r) = \begin{cases} b \log r + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3), \end{cases}$$

where b and c are constants.

Thus, we have the following for the fundamental solution:

Definition 3.1. The function

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & n \geq 3 \end{cases} \tag{6}$$

(where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$) is the *fundamental solution of the Laplace equation* as defined for $x \in \mathbb{R} \setminus \{0\}$ in [Eva10].

Now, let's consider an open set $U \subset \mathbb{R}^n$ and suppose u is a harmonic function within U.

Let's start with deriving the mean-value formula for the Laplace equation (similar to the mean value principle we proved in the previous section) which state that $u(x)$ equals both the average of u over the sphere $\partial B(x, r)$ and the average of u over the entire ball $B(x, r)$, given $B(x, r) \subset U$ [Eva10].

Theorem 3.1. (Mean-value formula for the Laplace equation) *If $u \in C^2(U)$ is harmonic, then*

$$u(x) = ⨍_{\partial B(x,r)} u dS = ⨍_{B(x,r)} u dy \tag{7}$$

for each ball $B(x,r) \subset U$.

The symbol $⨍$ is defined as the average of a function f over a ball or sphere with the following equations, respectively.

$$⨍_{B(x,r)} f dy = \frac{1}{\alpha(n) r^n} \int B(x,r) f dy$$

$$⨍_{\partial B(x,r)} f dS = \frac{1}{n\alpha(n) r^{n-1}} \int \partial B(x,r) f dS.$$

To prove the mean-value formulas, we need to use Green's formulas which we will state but not prove.

Theorem 3.2. (Green's formulas) *Let $u, v \in C^2(\overline{U})$ where $\overline{U}$ is the closure of U. Then*

(1) $\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial v} dS$,
(2) $\int_U Dv \cdot Du dx = -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial u}{\partial v} u dS$,
(3) $\int_U u\Delta v - v\Delta u dx = \int_{\partial U} u\frac{\partial u}{\partial v} - y\frac{\partial u}{\partial v} dS$.

Now we can continue to prove the mean-value formula.

Proof. Let

$$\phi(r) := ⨍_{\partial B(x,r)} u(y) dS(y) = ⨍_{\partial B(0,1)} u(x+rz) dS(z).$$

Then

$$\phi'(r) = ⨍_{\partial B(0,1)} Du(x+rz) \cdot z dS(z),$$

and by Green's formulas, we compute

$$\begin{aligned} \phi'(r) &= ⨍_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} dS(y) \\ &= ⨍_{\partial B(x,r)} \frac{\partial u}{\partial v} dS(y) \\ &= \frac{r}{n} ⨍_{\partial B(x,r)} \Delta u(y) dy = 0 \end{aligned}$$

Therefore ϕ is constant so

$$\phi(r) = \lim_{t \to 0} \phi(t) = \lim_{t \to 0} ⨍_{\partial B(x,r)} u(y) dS(y) = u(x).$$

□

An interesting result of the theorem we have just proven is that its converse is also true.

Theorem 3.3. (Converse to mean-value formula) *If $u \in C^2(U)$ satisfies*

$$u(x) = \fint_{\partial B(x,r)} u dS$$

for each Ball $B(x,r) \subset U$, then u is harmonic.

Proof. If $\Delta u \not\equiv 0$, there exists some ball $B(x,r) \subset U$ such that say $\Delta u > 0$ within $B(x,r)$. But for ϕ as defined above,

$$0 = \phi'(r) = \frac{r}{n} \fint_{\partial B(x,r)} \Delta u(y) dy > 0$$

which results in a contradiction. □

4. Applications

In general, PDEs have numerous applications in physics and engineering where there are more than two variables involved. Non-linear PDEs are mostly used for physics and mechanics. Examples of elliptic, hyperbolic, and parabolic include elasticity, wave propagation, and heat conduction, respectively. The harmonic functions are important in branches of physics such as electrostatics, gravitation, and fluid dynamics. Additionally, the Laplace equation describes the steady-state heat equation which does not depend on time.

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