COMPLEX DYNAMICS

EMMA CARDWELL, JOSHIKA CHAKRAVERTY, KATHERINE TAYLOR

Abstract. Complex dynamics is the study of the iterative behavior of of functions on Riemann surfaces. In this paper, we provide background on Riemann surfaces and their automorphisms and introduce basic concepts of complex dynamics. We introduce Julia and Fatou sets and discuss the Mandelbrot set as well as the classification of Fatou set components.

1. Introduction

Complex dynamics is the study of iterated mappings on complex spaces. More specifically, it's the study of what happens as we take $f(z)$, $f(f(z))$, $f(f(f(z)))$, …for holomorphic functions f on Riemann surfaces. [1] When discussing iterates of functions, there is some useful terminology we should go over.

Given some function f and a starting value w that f is iterated over, the orbit is simply the sequence of all possible outputs that result when we compute $f^n(w)$ for all values of n:

Definition 1.1 (Orbit)**.** Let f be a function from the set S to itself. For any $w \in S$, the **orbit** of w under f is the infinite sequence $\{f^n(w)\}_{n=0}^{\infty}$.

As an example, consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z + 2$. The orbit of 1 under f consists of the following:

$$\begin{aligned} f(1) &= 1 + 2 = 3 \\ f^2(1) &= 3 + 2 = 5 \\ f^3(1) &= 5 + 2 = 7 \\ f^4(1) &= 7 + 2 = 9 \\ &\vdots \end{aligned}$$

Thus, the orbit of 1 under f is the sequence $\{1, 3, 5, 7, 9, \ldots\}$.

An interesting question to consider is, what happens to the orbits of various functions when we start from different places? Can we have repeating orbits? If so, how frequently do they

Date: June 7, 2021.

[1]A note on notation: we denote by $f^n(z)$ the composition of f with itself n times (as opposed to the nth derivative, which we write as $f^{(n)}(z)$.)

repeat? To help us answer these questions, we have the concepts of periodic orbits and fixed points.

Definition 1.2 (Periodic Orbit)**.** Let f be a function from the set S to itself. For some $w \in S$, suppose n is the least positive integer such that $f^n(w) = w$. Then the sequence $\{w, f(w), f^2(w), \dots, f^n(w)\}$ is a **periodic orbit** (or **cycle**) of period n. (If there does not exist such an n, then the orbit of w is **aperiodic**.)

As an example, consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2 - 1$. The orbit of 0 under f is the sequence $\{0, -1, 0, -1, \dots\}$, so the sequence $\{0, -1\}$ is a periodic orbit of period 2.

Definition 1.3 (Periodic Point)**.** Let f be a function from the set S to itself, and let $w \in S$ such that $\{w, f(w), f^2(w), \dots, f^n(w)\}$ is a periodic orbit. Then w is a **periodic point** of f.

Returning to our previous example of $f(z) = z^2 - 1$, 0 and -1 are both periodic points of f.

Definition 1.4 (Fixed Point)**.** Let f be a function from the set S to itself. Then $w \in S$ is called a **fixed point** if $f(w) = w$.

Alternatively, a fixed point is a periodic point for an element of S with an orbit of period 1. Thus every fixed point is a periodic point, but not every periodic point is a fixed point.

As an example, consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$. The orbit of 0 under f is the sequence $\{0, 0, 0, \dots\}$, so the sequence $\{0\}$ is a periodic orbit of period 1 and 0 is a fixed point of f.

Definition 1.5 (Basin)**.** If $f : \mathbb{C} \to \mathbb{C}$ and $p \in \mathbb{C}$, then the **basin** of p is the set of points which iterate to p. Formally, the basin of p is $\text{Bas}(p) = \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = p\}$.

Definition 1.6 (Periodic Basin)**.** Suppose $f : \mathbb{C} \to \mathbb{C}$ and $p \in \mathbb{C}$ is periodic of period m, that is, $f^m(p) = p$. Then we can define

$$\text{Bas}(p) = \bigcup_{0\leq j<m} \{z \in \mathbb{C} : \lim_{n\to\infty} f^{nm+j}(z) = p\}.$$

The periodic basin can be considered as the basin of the entire orbit of p, since it is the union of the basins of each point in the orbit.

Given a function, how can we find orbits, periodic points, and basins? To answer this question, we shall look more closely at mappings on Riemann surfaces.

2. Automorphisms on Riemann Surfaces

We will consider dynamics on Riemann surfaces, particularly the Riemann sphere $\hat{\mathbb{C}}$. Before we begin our discussion of dynamics, we will provide a more in-depth introduction to Riemann surfaces, starting with the mappings between them. The most important kind of mapping is a **conformal automorphism**, which is a conformal mapping from a Riemann surface to itself. Recall that a conformal map is a bijective holomorphic function between open subsets of $\mathbb{C}$. [2]

Theorem 2.1 (Uniformization Theorem)**.** *Up to conformal isomorphism (the existence of a conformal map between two surfaces), there are only three simply connected Riemann surfaces: $\mathbb{C}$, the upper half plane $\mathbb{H}$, and the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$.*

The proof of this is beyond the scope of our paper, but can be found at [Abi81]. This is useful because it means that any simply connected Riemann surface can be nicely transformed into either $\mathbb{C}$, $\mathbb{H}$, or $\hat{\mathbb{C}}$. The properties of these three surfaces are very well known, allowing us to easily study other Riemann surfaces by comparison. The conformal automorphisms of each of these surfaces can be completely described. We will not be spending too much time with $\mathbb{C}$ or $\mathbb{H}$ after this, but for completeness we've included their automorphisms.

Theorem 2.2. *Every conformal automorphism f of $\mathbb{H}$ can written in the form*

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

Proof. Consider the function $g : \mathbb{H} \to \mathbb{D}$, where $g(z) = \frac{z-i}{z+i}$. Then for any automorphism f on $\mathbb{H}$, we can define an automorphism F on $\mathbb{D}$ given by $F = g \circ f \circ g^{-1}$.

In problem #7 from the Chapter 12 problem set of Simon's Complex Analysis book, we proved that all conformal automorphisms of $\mathbb{D}$ can be expressed in the form $g(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, where $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$. Thus, there must exist some $a, b \in \mathbb{C}$ with $|a|^2 - |b|^2 = 1$ such that we have $F(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ for all $z \in \mathbb{D}$. If we represent f as $f = g^{-1} \circ F \circ g$ and do some manipulation, we can show that f can be expressed as $f(z) = \frac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{R}$ and $ad - bc = 0$. A more complete explanation can be found at [Bra16c].

Proving that $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ is a conformal automorphism of $\mathbb{H}$ is left as an exercise for the reader. ∎

Theorem 2.3. *Every conformal automorphism of $\mathbb{C}$ is of the form*

$$f(z) = az + b$$

for $a, b \in \mathbb{C}, a \neq 0$.

Proof. Consider any function of the form $f(z) = az + b$ for $a, b \in \mathbb{C}$, $a \neq 0$. Since f is a linear map, f maps from $\mathbb{C} \to \mathbb{C}$. Furthermore, we have $f^{-1}(z) = \frac{z-b}{a}$, which is also a linear map. Thus, f is an automorphism of $\mathbb{C}$.

The other direction is more difficult to show, so please refer to [Bra16a] for an explanation. ∎

[2] Riemann surfaces and conformal mappings are introduced in Chapter 8 and Chapter 11 of Simon's Complex Analysis book, respectively.

Most of the dynamics we'll be talking about happens on the Riemann sphere.

Definition 2.4. The Riemann Sphere is the Riemann surface that is created when we glue two copies of the complex plane, $\mathbb{C}_1$ and $\mathbb{C}_2$, together using $f : \mathbb{C}_1 \to \mathbb{C}_2$ where $f(z) = \frac{1}{z}$ as a map. That is, every z in $\mathbb{C}_1$ is identified with $\frac{1}{z}$ in $\mathbb{C}_2$. The only two points that are left out are $z = 0$ in both copies. We consider $z = 0$ in $\mathbb{C}_1$ to be zero in the usual sense, and $z = 0$ in $\mathbb{C}_2$ to be the "point at infinity", sometimes referred to using the symbol ∞.

A nice way to visualize the Riemann sphere is as an actual sphere, where 0 is the bottom of the sphere and ∞ is the top. Intuitively, the Riemann sphere is just the complex numbers along with the point at infinity, which we can consider to be the point located infinitely far out in any direction. This makes the Riemann sphere better suited to the study of dynamics than the complex plane (at least in some ways). For example, if the orbit of a function diverges from the origin, we get to say instead that it converges to the point at infinity. It's worth noting that although we defined orbits, basins, etc. using functions $f : \mathbb{C} \to \mathbb{C}$, the exact same definitions can be used for functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The only difference is that ∞ is included as a point. For example, there is a such thing as the basin of ∞ (aka the set of points whose orbits go to infinity).

We now classify the automorphisms of $\hat{\mathbb{C}}$.

Theorem 2.5. *Every conformal automorphism f of $\hat{\mathbb{C}}$ can be expressed as a Möbius transformation of the form*

$$f(z) = \frac{az + b}{cz + d}$$

where $a, d, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Proof. A function is meromorphic in $\hat{\mathbb{C}}$ if and only if it is a rational map (for an explanation of this, please see [Bra16b]). Suppose f is an automorphism of $\hat{\mathbb{C}}$. Then f must be meromorphic, so it must be a rational map. This implies that f must map some point to ∞, so it must have a pole. Since f is a rational map with one pole, we can represent f as

$$f(z) = \frac{P(z)}{Q(z)}$$

for holomorphic functions P and Q on $\hat{\mathbb{C}}$. Since f is an automorphism, it must be injective. This implies that P and Q must be degree one polynomials with respect to z, so we have

$$f(z) = \frac{az + b}{cz + d}$$

for some $a, b, c, d \in \mathbb{C}$. We need $ad - bc \neq 0$ to guarantee the existence of f^{-1}.

Proving that $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ is an automorphism of $\hat{\mathbb{C}}$ is left as an exercise for the reader. ∎

Automorphisms of the Riemann sphere have relatively straightforward dynamical properties. For an automorphism g that isn't the identity, it either has two separate fixed points, or one with multiplicity 2. To see this, suppose our conformal automorphism is

$$f(z) = \frac{az+b}{cz+d}$$

and suppose z_0 is a fixed point. Then

$$z_0 = \frac{az_0+b}{cz_0+d}$$

or

$$cz_0^2 + (d-a)z_0 - b = 0.$$

As we know, a quadratic over $\mathbb{C}$ either has two solutions or one solution of multiplicity two, which finishes the proof whenever $c \neq 0$. In the case that $c = 0$, our fixed points will be $\frac{b}{d-a}$ and ∞. (We include ∞ as a fixed point because if z is infinitely large, so is $f(z) = \frac{az+b}{d}$.)

Definition 2.6 (Conformally conjugate)**.** Two automorphisms f, g are **conjugate** if there is another conformal map φ such that $g = \varphi^{-1} \cdot f \cdot \varphi$.

Conjugation is useful in dynamics because if $g(z) = \varphi^{-1}(f(\varphi(z)))$, then $g^n(z) = \varphi^{-1}(f^n(\varphi(z)))$. Thus, if we know the dynamics of g, we also know the dynamics of f.

Definition 2.7 (Affine Transformations)**.** A transformation of the form $g(z) = \lambda z + c$ where $\lambda, c \in \mathbb{C}$ and $\lambda \neq 0$.

These are a very simple kind of transformation. As it turns out, conformal automorphisms of the Riemann sphere are closely related to affine transformations.

Proposition 2.8. *Every conformal automorphism of $\hat{\mathbb{C}}$ with two distinct fixed points is conformally conjugate to an automorphism of the form $f(z) = \lambda z$, $\lambda \in \mathbb{C}$.*

Proof. Suppose g is a conformal automorphism with two fixed points w_1 and w_2. We can move one of them to 0 and the other to infinity by conjugation with the map

$$\varphi(z) = \frac{z - w_1}{z - w_2}.$$

Consider $f(z) = \varphi \cdot g \cdot \varphi^{-1}(z)$, which is a conformal conjugate of g. Let

$$f(z) = \frac{az+b}{cz+d}.$$

Since 0 is a fixed point of f, we have

$$f(0) = 0 = \frac{0+b}{0+d}$$

which implies that $b = 0$, $d \neq 0$. We also have that ∞ is a fixed point, so

$$f(\infty) = \infty = \frac{a\infty + b}{c\infty + d}$$

which can only be true if $c = 0$. (If the reader is uncomfortable with using ∞ as a number, this can also be easily proved by considering $\lim_{z\to\infty} f(z)$.) Thus,

$$f(z) = \frac{az+0}{0+d} = \frac{a}{d}z$$

which completes the proof. ■

Proposition 2.9. *Every conformal automorphism of $\hat{\mathbb{C}}$ with one fixed point of multiplicity two is conformally conjugate to an automorphism of the form $f(z) = z + c$, $c \in \mathbb{C}$.*

Proof. Suppose that we have a single fixed point w_1. This time we conjugate by the conformal automorphism

$$\varphi(z) = \frac{1}{z - w_1}$$

so that $f(z) = \varphi \cdot g \cdot \varphi^{-1}(z)$ has its fixed point at ∞. We have already seen that if ∞ is a fixed point, $c = 0$, so we must have

$$f(z) = \frac{az + b}{cz + d} = \frac{a}{d}z + \frac{b}{d}$$

for some $a, b, d \in \mathbb{C}$. Now suppose that z is a fixed point of $f(z)$. We have

$$z = \frac{a}{d}z + \frac{b}{d}$$

or

$$z = \frac{b}{d(1 - a/d)}$$

Since we already know $z = \infty$, we must have $\frac{a}{d} = 1$, and so $f(z) = z + \frac{b}{d}$, which completes the proof in the case with one fixed point. ■

This will help us to show that for automorphisms of $\hat{\mathbb{C}}$, there are three basic kinds of behavior.

(1) An **elliptic** transformation $g(z)$ is one which is conformally conjugate to $f(z) = \lambda z$ with $|\lambda| = 1$. This means that f is a rotation of the Riemann sphere. Orbits of f do not tend toward any specific place on the sphere, and therefore, neither do the orbits of g. We say that the two fixed points of g are neither attractive nor repelling.
(2) A **hyperbolic** transformation $g(z)$ is conjugate to $f(z) = \lambda z$ with $|\lambda| \neq 1$. If $|\lambda| > 1$, then all orbits of f tend towards ∞. Every point except for 0 is in the basin of ∞. In this case we say that ∞ is an attractive fixed point and 0 is a repelling fixed point. If $|\lambda| < 1$, the opposite is true. In either case, $g(z)$ will have one attractive fixed point and one repelling fixed point.
(3) A **parabolic** transformation g has one fixed point with a multiplicity of two. $g(z)$ is conjugate to $f(z) = z + c$. In general, orbits of f leave ∞ in one direction, go around the sphere, and come back on the other side. This is easier to see with an example. Take $f(z) = z + 1$, and consider a point with a large negative real component. The orbit of this point starts near ∞ on the left side of the complex plane, eventually reaches the imaginary axis, and then continues out towards ∞ in the positive real direction.

We characterized the fixed points of these automorphisms as "attracting" or "repelling". We proceed to formally define these terms.

Definition 2.10 (Attractive fixed point)**.** $z_0 \in \hat{\mathbb{C}}$ is an **attractive fixed point** of $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ if $f(z_0) = z_0$ and there exists a neighborhood U of z_0 such that for every $w \in U$, the orbit of w tends to z_0. Equivalently, z_0 is attractive if for some neighborhood U of z_0, every point in U is in the basin of z_0.

Any point which is sufficiently close to an attractive fixed point of f will converge to the attractive fixed point under repeated application of f. For instance, 0 is an attractive fixed point of $f(z) = z^2$.

Definition 2.11 (Repelling fixed point)**.** $z_0 \in \hat{\mathbb{C}}$ is a **repelling fixed point** of $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ if $f(z_0) = z_0$ and there exists a deleted neighborhood U of z_0 such that for every $w \in U$, the orbit of w will eventually leave U.

Any point which is sufficiently close to a repelling fixed point of f will have its orbit tend away from the repelling fixed point. For example, 0 is a repelling fixed point of $f(z) = \sqrt{z}$.

Just like we can have attracting and repelling fixed points, we can also have attracting and repelling periodic points. The idea behind attracting periodic points is that as we take iterates of f, they tend towards a specific repeated sequence of points. Each point in the sequence being approached would be considered an attracting periodic point.

Definition 2.12 (Attractive periodic point)**.** Suppose z_0 is a periodic point of order n (that is, n is the smallest positive integer such that $f^n(z_0) = z_0$). Then z_0 is an attractive periodic point if there exists a neighborhood U of z_0 such that for every $w \in U$, the orbit of w under f^n (not under f) tends to z_0. Equivalently, z_0 is attractive if for some neighborhood U of z_0, every point in U is in the n-periodic basin of z_0.

For example, consider the logistic function defined by $x_{t+1} = 0.5x_t(1 - x_t)$. By plugging in various values for x_0, we see that 0 is an attracting periodic point. As another example, consider $x_{t+1} = 2x_t(1 - x_t)$. Then $\frac{1}{2}$ is an attracting periodic point.

Definition 2.13 (Repelling periodic point)**.** If z_0 is a periodic point of order n, then z_0 is a repelling periodic point if there exists a deleted neighborhood U of z_0 such that for every $w \in U$, the orbit of w under f^n will eventually leave U.

The iterative behavior of a fixed point or periodic point has a lot to do with the value of the derivative of f at that point. Suppose z_0 is a fixed point and $|f'(z_0)| < 1$. Then for all points w which are sufficiently close to z_0, we have

$$f'(z_0) \approx \frac{f(w) - f(z_0)}{w - z_0}$$

from which we derive

$$f(w) - f(z_0) = f(w) - z_0 \approx (w - z_0)f'(z_0).$$

Thus, we have

$$|f(w) - z_0| \approx |(w - z_0)f'(z_0)| < |w - z_0|$$

meaning that $f(w)$ is closer to z_0 than w is. Similarly, we can prove that $f^2(w)$ is even closer to z_0, etc. In general, the distance to z_0 will decrease by a factor of $f'(z_0)$ with each iteration.

We conclude that for every w which is close enough to z_0, the orbit of w will converge to z_0, and so z_0 is an attractive fixed point. An analogous argument will show that if $f'(z_0) > 1$, z_0 is a repelling fixed point.

Our goal here is not to provide a rigorous proof of the relationship between the derivative of f and the iterative behavior of f, but rather to provide some justification for the following classification of periodic point behavior.

Though we have only defined attracting and repelling points so far, the reader may have noticed there are also other types of behavior. For instance, the parabolic fixed points of $f(z) = z + c$ do not fit into either of these categories. We will list some other kinds of periodic points, defined using the magnitude of the derivative at that point. Let $g(z)$ be an analytic automorphism. At the periodic point z_0, define the multiplier as $\gamma = |g'(z_0)|$. It is a fact, beyond the scope of this paper to prove, that γ takes the same value for all points in the orbit of z_0. (Thus, all points in the same periodic orbit display the same dynamic behavior.) These are the types of periodic points:

(1) If $\gamma = 0$, then z_0 is **super-attracting**.
(2) If and only if $0 < \gamma < 1$, then z_0 is **attracting**.
(3) If and only if $\gamma > 1$, then z_0 is **repelling**.
(4) If $g'(z_0)$ is a root of unity, then z_0 is **rationally indifferent**.
(5) If $\gamma = 1$ and $g'(z_0)$ is not a root of unity, then z_0 is **irrationally indifferent**.

Parabolic fixed points are rationally indifferent, as the derivative of $f(z) = z + c$ is 1 everywhere. We will not go into too much detail regarding rationally or irrationally indifferent fixed points.

3. Fractals

Fractals are loosely defined as objects that display self-similarity. This could mean that the exact same structure shows up several times, or that similar structures appear. In this section, we will describe some fractals that appear from recursive functions on complex numbers on the complex plane. Informally, the way we create a fractal from a recursive function is by coloring the points of the complex plane different colors depending on the iterative behavior of the function when we start from that point. Fractals are useful in the study of complex dynamics because we can use fractals to understand the behavior of functions as we iterate them, and because they look cool. As a simple example, consider the function $f(z) = z^2$. If $|z| < 1$ then $f^n(z)$ approaches 0 as n increases, and if $|z| > 1$ then $f^n(z)$ approaches ∞. If $|z| = 1$ then $f^n(z)$ will stay in the unit circle, and different values of z in the unit circle will produce rather different behavior. For instance, $z = e^{\pi i/2^k}$ will eventually stop at the fixed point $z = 1$, while other rational roots of unity will cycle forever, and irrational roots of unity will have even more erratic behavior. A good visual representation of this would be the complex plane with the unit disk, the complement of the unit disk, and the unit circle all shaded different colors. It's useful to have a formal definition for these different regions with different behavior, so that's what we'll look at next.

Recall that a **normal family** is a family of holomorphic functions $f : \Omega \to \mathbb{C}$ for some open subset $\Omega \subset \mathbb{C}$ where every sequence $f_1, f_2, \ldots$ in the family contains a subsequence $f_{k_1}, f_{k_2}, \ldots$ that converges uniformly on all compact subsets of Ω. Additionally, Montel's Theorem tells us that any family of holomorphic functions that are all uniformly bounded on compact subsets is a normal family. These are covered in Chapter 12 of Simon's Complex Analysis book, so we shall skip a more thorough introduction of this topic in our paper.

3.1. **Fatou Sets and Julia Sets.**

Definition 3.1. Let S be a Riemann surface, and let $f : S \to S$ be a non-constant holomorphic mapping. Fixing some point $z_0 \in S$: If there exists some neighborhood U of z_0 so that the sequence of iterates f^n restricted to U forms a normal family, then z_0 is a normal point and belongs to the **Fatou set** of f. Otherwise, if no such neighborhood exists, then z_0 belongs to the **Julia set** of f.

Note that the Fatou set is open, and that the Fatou and Julia sets are complements. Informally, the Fatou set consists of all points where the dynamics of f is relatively tame, and the Julia set consists of those points where the dynamics of f is more interesting. Consider our example with $f(z) = z^2$ where our Riemann surface is the Riemann sphere. As we will show, the Fatou and Julia sets capture the different behavior we found for points inside, outside, and on the unit circle.

For any point z with $|z| < 1$, choose a neighborhood U of z such that for all $w \in U$, $|w| < 1$. Then the sequence f^n converges uniformly to the function $g(z) = 0$, so the unit disk is contained in the Fatou set of f. Similarly, for any z with $|z| > 1$, the sequence f^n converges uniformly to the function $h(z) = \infty$, so the region outside the unit circle is also contained in the Fatou set of f. The interior and exterior of the unit circle, respectively, are called **components** of the Fatou set. There is a lot to discuss regarding the characteristics of different components of a Fatou set, which we leave for the next section.

Returning to the unit circle, if we choose any z with $|z| = 1$, then any neighborhood of z contains some points for which f^n goes to 0 and other points for which f^n goes to ∞, so the sequence f^n cannot form a normal family. Thus, the Julia set of $f(z) = z^2$ is the unit circle.

Functions f and p are conjugates under φ if $f(z) = \varphi^{-1}(p(\varphi(z)))$. Under iteration, we end up with $f^n = \varphi^{-1} \circ p^n \varphi$. This is because the φ and φ^{-1} that are next to each other cancel out.

For example,

$$\begin{aligned} f^2 &= (\varphi^{-1} \circ p \circ \varphi) \circ (\varphi^{-1} \circ p \circ \varphi) \\ &= \varphi^{-1} \circ p \circ (\varphi \circ \varphi^{-1}) \circ p \circ \varphi \\ &= \varphi^{-1} \circ p \circ p \circ \varphi \\ &= \varphi^{-1} \circ p^2 \circ \varphi. \end{aligned}$$

This shows that we only have one ϕ and one ϕ^{-1}. We can look at f^3 next:

$$\begin{aligned}
f^3 &= f^2 \circ f \\
&= \varphi^{-1} \circ p^2 \circ \varphi \circ (\varphi^{-1} \circ p \circ \varphi) \\
&= \varphi^{-1} \circ p^2 \circ (\varphi \circ \varphi^{-1}) \circ p \circ \varphi \\
&= \varphi^{-1} \circ p^2 \circ p \circ \varphi \\
&= \varphi^{-1} \circ p^3 \circ \varphi
\end{aligned}$$

Following this pattern, we will end up with $f^n = \varphi^{-1} \circ f^n \circ \varphi$. This conjugation is a useful notion because if we have a complicated f^n, if we can find a conjugate, we can look at the behavior of the Julia Set of p instead.

One function that we can look at with conjugation is $f(z) = z^2 - 2$. For $f^2(z) = (z^2-2)^2-2$ and then we have $((z^2-2)^2-2)^2-2$. The constant subtracting of 2 makes it difficult to see what the Julia set may look like. We can conjugate with $\varphi(w) = w + \frac{1}{w}$, $\varphi : w : |w| > 1 \to \mathbb{C}\ [-2, 2]$. This ϕ is taking the space outside a unit disk and mapping it onto the complex plane except for the line on the real axis from $[-2, 2]$. We don't want to include that part because $f(z) : [-2, 2] \to [-2, 2]$. Looking at $\varphi^{-1} \circ f \circ \varphi$. Taking a point outside the unit circle, φ moves it to $\mathbb{C}\ [-2, 2]$, then f moves it to another part of $\mathbb{C}\ [-2, 2]$, and φ moves it back to the outside of the unit circle. To understand f, we want to get a clearer look into what the function $\varphi^{-1} \circ f \circ \varphi$ is. Plugging in, we get:

$$\begin{aligned}
f(\varphi(w)) &= \varphi(w)^2 - 2 \\
&= (w + \frac{1}{w})^2 - 2 \\
&= w^2 + \frac{1}{w^2} + 2w\frac{1}{w} - 2 \\
&= w^2 + \frac{1}{w^2} + 2 - 2 \\
&= w^2 + \frac{1}{w^2} \\
&= \varphi(w^2).
\end{aligned}$$

This means that $\varphi^{-1} \circ f \circ \varphi = w^2$. We already know what the Julia set of w^2 is. This means that on $\mathbb{C}\ [-2, 2]$, $f(z) = z^2 - 2$ behaves like w^2 and $\lim_{n\to\infty} f^n(z) = \infty$ for $z \in \mathbb{C}\ [-2, 2$. This leaves the Julia set to be $[-2, 2]$.

Theorem 3.2. *Let $p(z) = az^2 + bz + d$ with $a, b, d \in \mathbb{C}$ and $f(z) = z^2 + c$ with $c \in \mathbb{C}$. For each triple of constants (a, b, d), there is exactly one c such that $p(z)$ and $f(z)$ behave the same under iteration.*

Define $c = ad + \frac{b}{2} - \frac{b}{2}^2$ and choose $\varphi(z) = az + \frac{b}{2}$. We want to show that $p(z) = \varphi^{-1}(f(\varphi(z)))$ for all z.

The dynamics of the family of functions $f_c(z) = z^2 + c, c \in \mathbb{C}$ is extremely complicated and interesting, as we will see when we begin to talk about the Mandelbrot set. For any f_c, we can find its Julia set. We already know that for $c = 0$ the Julia set is the unit circle, but that is by far the simplest case.

Definition 3.3 (Filled-in Julia Set)**.** Let $f(z)$ be the rational function $\frac{P(z)}{Q(z)}$ where P, Q are polynomials with no common divisors and $z \in \hat{\mathbb{C}}$. Then the **filled-in Julia Set**, denoted $K(f)$, is the set of points z that do not approach ∞ when $f(z)$ is repeatedly applied.

Proposition 3.4. *The Julia set $J(f)$ is the boundary of $K(f)$.*

The proof of this proposition is beyond the scope of our paper, but can be found in [Mil00]. Consider our earlier example of $f(z) = z^2$. We can see immediately that the points which do not approach ∞ are exactly those points on or inside the unit circle, so $K(f) = \{z \in \mathbb{C} : |z| \leq 1\}$. As claimed, the boundary of this set is the Julia set $J(f)$.

The Fatou set is open and the Julia set is compact. They are complements of each other for each particular z_0. Additionally, if p is an attracting or super-attracting periodic point, then $\text{Bas}(p) \subset \mathcal{F}$, the Fatou set. If p is a repelling periodic point, then $p \in \mathcal{J}$, the Julia set.

Now, we will look at the Mandelbrot set.

3.2. Mandelbrot Set.

Definition 3.5 (Mandelbrot Set)**.** The Mandelbrot Set is the set of complex numbers c such that after iterating $f_c(z) = z^2 + c$ starting from 0, the orbit does not tend to infinity. In other words, it is the set of c such that 0 is contained in the filled-in Julia set of f_c.

We can prove some interesting things about the types of numbers that belong to the Mandelbrot Set. Specifically:

Proposition 3.6. *For any c and $f_c(z) = z^2 + c$, if $|z_n| > 2$ for any n, then c will not belong to the Mandelbrot set.*

Proof. We can prove this using some casework. First, consider the case where $|c| \leq 2$. Then we have:

$$z_{n+1} = z_n^2 + c$$

From a corollary of the Triangle Inequality Theorem, we see that

$$|z_{n+1}| \geq |z_n^2| - |c|$$

Now, since $|z_n| > 2$, we can say

$$|z_{n+1}| \geq |z_n^2| - |c|$$
$$|z_{n+1}| > 2|z_n| - |c|$$

Furthermore, we assumed that $|c| \leq 2$, so we have:

$$\begin{aligned}|z_{n+1}| &> 2|z_n| - 2\\ |z_{n+1}| - 2 &> 2|z_n| - 4\\ |z_{n+1}| - 2 &> 2(|z_n| - 2)\end{aligned}$$

This shows that the distance between z_n and 2 increases as n increases, so z_n approaches infinity as n approaches infinity.

Next, consider the case where $|c| > 2$. Then we have:

$$\begin{aligned}&|z_0| = |0| = 0\\ &|z_1| = |f_c(z_0)| = |0^2 + c| = |c|\\ &|z_2| = |f_c(z_1)| = |c^2 + c| \geq ||c^2| - |c|| > |c|\\ &|z_3| \geq |z_2|^2 - |c|\\ &|z_3| > |c||z_2| - |c|\\ &|z_3| > |z_2| + \big(|c| - 1\big)|z_2| - |c|\\ &|z_3| > |z_2| + \big(|c| - 1\big)|c| - |c|\\ &|z_3| > |z_2| + \big(|c|^2 - 2|c|\big)\end{aligned}$$

Since $|c| > 2$, we know that $|c|^2 - 2|c| > 0$ must be true. This implies that $|z_n| > |c|$ for all $n \geq 2$. Now, using induction, we get:

$$\begin{aligned}&|z_{n+1}| = |z_n^2 + c|\\ &|z_{n+1}| \geq |z_n|^2 - |c|\\ &|z_{n+1}| > |c||z_n| - |c|\\ &|z_{n+1}| > |z_n| + \big(|c| - 1\big)|z_n| - |c|\\ &|z_{n+1}| > |z_n| + \big(|c|^2 - 2|c|\big)\end{aligned}$$

This implies that $|z_n|$ approaches infinity as n approaches infinity.

In both the $|c| \leq 2$ and the $|c| > 2$ cases, we saw that the orbit of z_n approached infinity if $|z_n| > 2$. Thus, we have shown that for any value of c, if there exists some n such that $|z_n| > 2$, then c is not in the Mandelbrot set. ■

One way the Mandelbrot set can be examined is in terms of the shapes that make it up. For all values of c in the central cardiod of the Mandelbrot set, f_c has an attracting fixed point. The other circly bits are for values of c with attracting cycles of various other period lengths.

Another interesting connection between the Mandelbrot Set and Julia Sets: taking any point on the complex plane, if you draw the Julia Set using that point as c and get a disconnected set that looks like a bunch of dots, then we are not in the Mandelbrot Set. However, if you

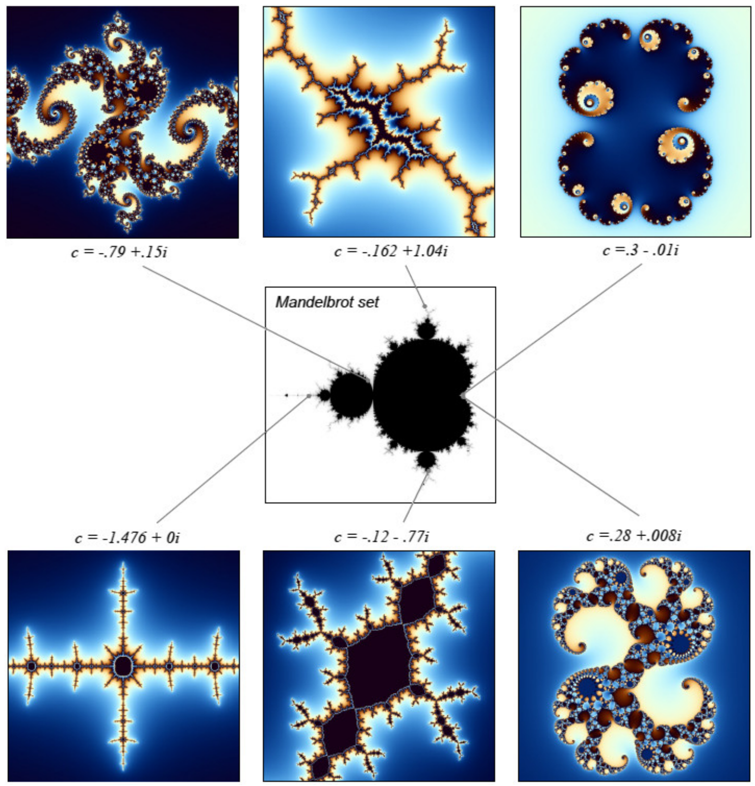

Figure 1. Some Julia sets for different values of c, along with the location of c in the Mandelbrot set. The complement of each Julia set is colored according to how many iterations it takes for $f^n(z)$ to escape a certain bound [Sim11].

do get a connected region, then that point is in the Mandelbrot Set. For this reason, the Mandelbrot set is referred to in certain circles as the "does-it-dust diagram".

If we only consider $c \in \mathbb{R}$, then $f_c(z) = z^2 + c$ is equivalent to the logistic map $x_{n+1} = rx_n(1 - x_n)$ by the change of variables $x_n = -\frac{1}{r}z + \frac{1}{2}$. We have

$$\begin{aligned}
x_{n+1} &= rx_n(1 - x_n) \\
&= r\left(-\frac{1}{r}z + \frac{1}{2}\right)\left(\frac{1}{r}z + \frac{1}{2}\right) \\
&= -\frac{1}{r}z^2 + \frac{r}{4} \\
&= -\frac{1}{r}\left(z^2 + \frac{r}{2} - \frac{r^2}{4}\right) + \frac{1}{2} \\
&= \frac{1}{r}f(z) + \frac{1}{2}
\end{aligned}$$

as desired. (So, the correspondence between the constants r and c is $c = \frac{r}{2} - \frac{r^2}{4}$.)

The usual way the behavior of the logistic map is shown is with the bifurcation diagram. In the bifurcation diagram, r ranges between roughly 2 and 4 along the horizontal axis. The

Figure 2. Bifurcation vs. The Brot [Lay08].

vertical axis shows the attractive periodic points for that value of r. It's called a bifurcation diagram because as r increases, it goes from having a single fixed point to having a 2-cycle, to a 4-cycle, and so on.

This provides us with a satisfying picture of the relationship between the Mandelbrot set and the bifurcation diagram. Since the quadratic and logistic maps are equivalent, they have the same dynamic behavior. The main cardioid consists of values of c with a single fixed point, so it corresponds to the values of r for which the logistic map has a single fixed point. The circle behind the cardioid consists of values of c with a 2-cycle, so it corresponds to the values of r where the logistic map has a 2-cycle, and so on.

4. Classification of Fatou Sets

Recall that the Fatou set of a complex function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined as the set of points where the sequence of iterates $\{f^{(n)}(z)\}$ converges uniformly to some limiting function $g(z)$. A Fatou set is not necessarily a connected set. For example, we saw that the Fatou set of $f(z) = z^2$ is the entire complex plane except for the unit circle, so it's split into two **components** which are the interior and exterior of the unit circle respectively. More formally, since Fatou sets are open sets, we can express every Fatou Set as a discrete union of connected sets called **Fatou components**.

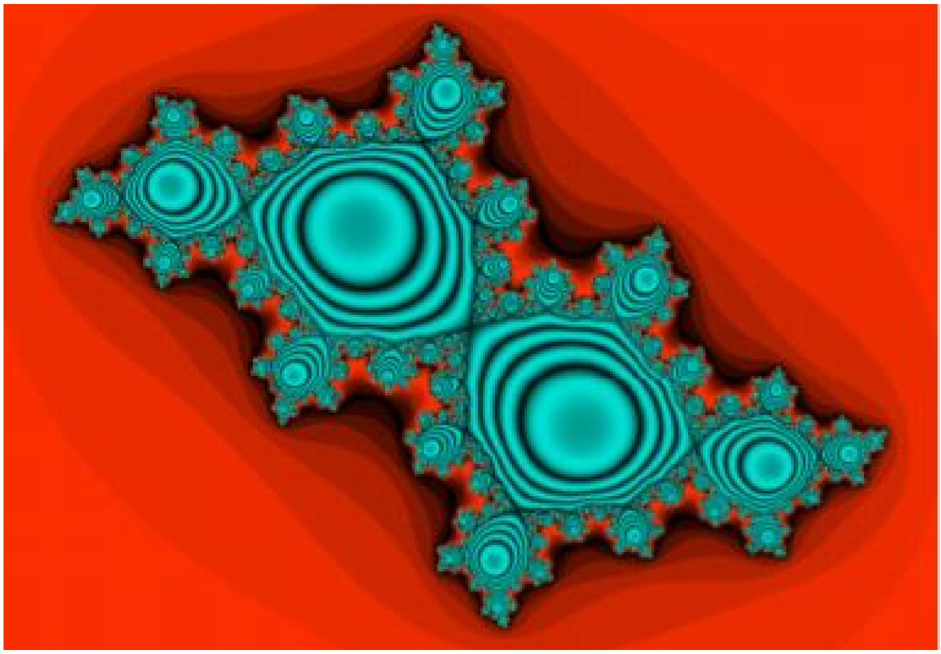

Figure 3. The Julia set of a polynomial which has a Siegel disk with a rotation number of the golden mean: the Siegel disk and its preimages are shown in blue; the light-dark bands indicate orbits within the Siegel disk [Sut14a].

It is an interesting fact that if U is a Fatou component of the function f, then $f(U) = \{f(x) : x \in U\}$ is also a Fatou component. We omit the proof, which can be found in [Lli18].

Definition 4.1. A Fatou component $U = f(U)$ is a rotation domain if there exists a conformal map $\varphi : U \to V$ conjugating f to an irrational rotation. There means that there is a $\lambda = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R}\mathbb{Q}$ and $\varphi(f(z)) = \lambda\varphi(z)$ for all $z \in U$.

In the previous definition φ is a conformal map. If f is a rational map with a rotation domain U, then it is a **Siegel Disk** if U is simply connected and an **Arnold-Herman Ring** if U is double connected. A Siegel disk will always have a fixed point at $z = \varphi^{-1}(0)$ and and Arnold-Herman disk has no fixed points.

The Julia set of polynomials are complicated even though they are rotation domains. Figure 3 is an example.

Arnold-Herman rings cannot occur on polynomials and only occur on rational maps with odd degree of at least three. When we have a degree value that works, we can construct Blaschke products to set the unit circle to itself. An example is $f = e^{2\pi it}z^2\frac{z-4}{1-4z}$. If we set t 0.615732, we get a rotation number approximately equal to the golden mean and we end up with the rational map seen in Figure 4.

Then for a periodic component U, there are only four possibilities:

(1) U contains an attracting or super attracting point

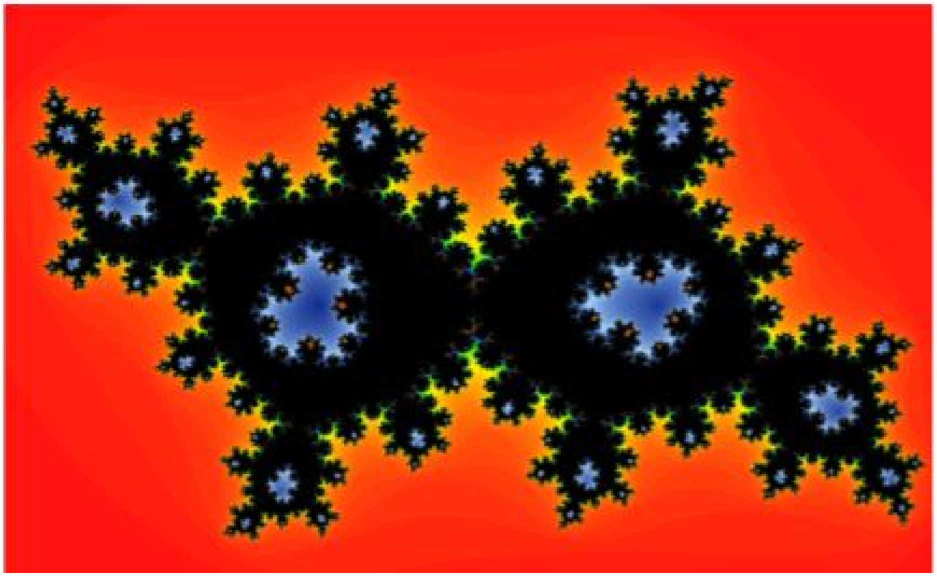

Figure 4. A rational map with a Hermann ring with a golden-mean rotation: Bas(∞) is in reds and yellows, Bas(0) in shades of blue, with the Hermann ring and its preimages shown in black [Sut14b].

(2) U is parabolic
(3) U is a Siegel disc
(4) U is a Herman ring

The Fatou set has 0, 1, 2 or infinite components. Dennic Sullivan proved that every Fatou Component is eventually periodic in `https://www.math.stonybrook.edu/~dennis/publications/PDF/DS-pub-0059.pdf`.

Through the Julia and Fatou sets, we can also tell whether a function is conjugate to a polynomial or not. From Montel's Theorem, we can also define exceptional points in the Julia Set. If z is in the Julia Set, exceptional points are the points in $\mathcal{E}_U$ where $\mathcal{E}_U = \hat{\mathbb{C}} \cup_{n>0} f^n(U)$ in a neighborhood U of z. $\mathcal{E}_U$ has at most two points. For example, in a polynomial ∞ is an exceptional point and a super attracting point. We also have Bas(∞) is a subset of the Fatou Set and the Julia Set is in a bounded region of $\mathbb{C}$.

Theorem 4.2. *Suppose z is in the Julia Set and define $\mathcal{E}_z = \cup \mathcal{E}_U$ where U ranges over all neighborhoods of z. Then*

(1) *If $\mathcal{E}_z$ has only one point, then f is conjugate to a polynomial*
(2) *If $\mathcal{E}_z$ contains two points, then f is either conjugate to z^d or $\frac{1}{z^d}$ where d us the degree of f*

This conjugation does not depend on the choice of z. This theorem can be proved from with Mobius Transformations that move $\mathcal{E}_U$ to ∞ when there is only one point and to $0, \infty$ when there are two points.

References

[Abi81] William Abikoff. The uniformization theorem. *The American Mathematical Monthly*, 88(8):574–592, 1981. URL: http://www.jstor.org/stable/2320507.

[Bra16a] Tai-Danae Bradley. Automorphisms of the complex plane, Jun 2016. URL: https://www.math3ma.com/blog/automorphisms-of-the-complex-plane.

[Bra16b] Tai-Danae Bradley. Automorphisms of the riemann sphere, Jul 2016. URL: https://www.math3ma.com/blog/automorphisms-of-the-riemann-sphere.

[Bra16c] Tai-Danae Bradley. Automorphisms of the upper half plane, Jun 2016. URL: https://www.math3ma.com/blog/automorphisms-of-the-upper-half-plane.

[Lay08] Georg-Johann Lay. *Diagram showing the connexion between Verhulst dynamic and Mandelbrot set*. Apr 2008. URL: https://commons.wikimedia.org/wiki/File:Verhulst-Mandelbrot-Bifurcation.jpg.

[Lli18] Robert Florido Llinàs. Classification of periodic fatou components for rational maps, 2018. URL: http://diposit.ub.edu/dspace/bitstream/2445/122538/2/memoria.pdf.

[Mil00] John Milnor. *Dynamics in one complex variable: introductory lectures*. Vieweg, 2000.

[Sim11] Karl Sims. *Six Julia sets and their corresponding locations in the Mandelbrot set*. 2011. URL: https://www.karlsims.com/julia.html.

[Sut14a] Scott Sutherland. *Julia set of a polynomial with a Siegel disk*. 2014. URL: http://www.math.stonybrook.edu/~scott/Papers/India/Fatou-Julia.pdf.

[Sut14b] Scott Sutherland. *Rational map with a Hermann ring*. 2014. URL: http://www.math.stonybrook.edu/~scott/Papers/India/Fatou-Julia.pdf.

Euler Circle, Palo Alto, CA 94306