THE j INVARIANT

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ABSTRACT. In this paper, we explore the j-invariant and discuss some of its properties which we use later as we explore other related topics such as modular functions and elliptic curves. We also discuss how the j-invariant is connected to other results related to Weierstrass function and Eisenstein series.

INTRODUCTION

The Klein's j-invariant, named after Felix Klien, is responsible for many seemingly mysterious results in Number Theory and other fields. Though we will not discuss many of them in this paper, we will see many properties of the function and why it works very nicely in many related areas.

1. MODULAR FORMS

We begin by defining by defining a modular group:

Definition 1.1. The modular group Γ is the group of linear fractional transformations of the upper half of the complex plane, which have the form

$$z \to \frac{az+b}{cz+d}$$

where a, b, c, d are integers and $ad - bc = 1$.

The modular group can be shown to be generated by the two transformations

$$S : z \to -1/z,$$

$$T : z \to z + 1.$$

so that every element in the modular group can be represented (in a non-unique way) by the composition of powers of S and T.

A modular form is an analytic function on $\mathbb{H}$ satisfying a functional equation with respect to the group action of the modular group. More generally, we give the following definition:

Definition 1.2. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Given a subgroup $\Gamma \in SL(2, Z)$ of finite index, we define a modular form of level Γ and weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that:

1. (automorphy condition) For any $\gamma \in \Gamma$ there is the equality $f(\gamma(z)) = (cz + d)^k f(z)$
2. (growth condition) For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ the function $(cz + d)^{-k} f(\gamma(z))$ is bounded for $\mathrm{im}(z) \to \infty$.

Additionally, we say it is in cusp form if for any γ in $SL_2(\mathbb{Z})$, the function $(cz+d)^{-k} f(\gamma(z)) \to 0$ as $\mathrm{im}(z) \to \infty$.

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For modular forms for the modular group $SL(2,\mathbb{Z})$, we have the following definition:

Definition 1.3. A modular form of weight k for the modular group $SL(2,\mathbb{Z})$ is a complex valued function f on $\mathbb{H}$ satisfying:
1. f is holomorphic on $\mathbb{H}$.
2. For any $z \in \mathbb{H}$, and any matrix $SL(2,\mathbb{Z})$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$
3. f is holomorphic as $z \to i\infty$.

Equivalently, we may define a modular form in terms of lattices:

Definition 1.4. A modular form is a function F from the set of lattices of $\mathbb{C}$ to $\mathbb{C}$ which satisfies:
1. Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Then $F(\Lambda)$ is an analytic function of z.
2. Let α be a non-zero complex number. Then $F(\alpha\Lambda) = \alpha^{-k}F(\Lambda)$.
3. The absolute value of $F(\Lambda)$ remains bounded above as long as the absolute value of the smallest non-zero element in Λ is bounded away from 0.

If we specify the weight of a modular form $k = 0$, then by Liouville Theorem, the only modular forms are constant functions.

2. Elliptic Curves and the j-function

Definition 2.1. An elliptic curve E is an equation of the form $y^2 = x^3 + ax^2 + bx + c$.

These elliptic curves may be put in *Weierstrass Form* $y = x^3+ax+b$ through substitutions preserving rational points.

Lemma 2.2. *E does not have any self intersections or cusps.*

Sketch of proof. We may see this by defining $F(x,y) = x^3 + ax^2 + bx + c - y^2$ and checking if $\Delta F = \vec{0}$ when $F(P) = 0$ at every point P. ■

We call the property of the above lemma as nonsingularity.

Definition 2.3. Consider an elliptic curve E over $\mathbb{Q}$ and let $y^2 = x^3+ax+b$ be its Weierstass form. Then its j -invariant is given by $j(E) = \frac{4a^3}{4a^3+27b^2}$.

However, the above definition of the j-invariant does not tell us much about its use or importance. To motivate the purpose of the function, we first need to reinterpret elliptic curves, for which, we need to define elliptic functions.

Let Λ be a lattice and let

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

Also define the Eisenstein series of weight $2k$ as

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}\setminus(0,0)} \frac{1}{(m+n\tau)^2k}.$$

It can be shown that we have the following differential equation:

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6$$

where G_4 and G_6 are the Eisenstein series with weight 4 and 6, respectively. Thus, we find that $(\wp(z), \wp'(z))$ lies on the curve defined by the equation

$$E : y^2 = 4x^3 - g_2 - g_3$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$. With this, define $E_\Lambda : y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$. Then there is an isomorphism between $\mathbb{C}/\Lambda$ and $E_\Lambda(\mathbb{C})$ as groups.

Now we want to look at homothetic lattices. We present the following definition.

Definition 2.4. We say that two lattices Λ and Λ' are homothetic if $\Lambda = \omega\Lambda'$ for $\omega \in \mathbb{C}\backslash\{0\}$.

Then we have the following lemma.

Lemma 2.5. *The Complex Tori $\mathbb{C}/\Lambda \cong E_\Lambda$ and $\mathbb{C}/\Lambda' \cong E'_\Lambda$ are isomorphic if and only if the two lattices Λ and Λ' are homothetic.*

We will see a proof of this later.

Now we define the j-function, which turns out to be very useful for such things.

The $j-$ function is a modular function of weight zero for SL(2, Z) defined on $\mathbb{H}$. It is the unique such function which is holomorphic away from a simple pole at the cusp which satisfies

$$j(e^{2\pi i/3}) = 0 \quad \text{and} \quad j(i) = 1728 = 12^3.$$

We give the formal definition as follows:

Definition 2.6. The $j-$ function is function defined on $\mathbb{H}$,

$$j(\tau) = \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2},$$

where

$$g_2(\tau) = 60 \sum_{(m,n)\neq(0,0)} (m + n\tau)^{-4},$$
$$g_3(\tau) = 140 \sum_{(m,n)\neq(0,0)} (m + n\tau)^{-6}.$$

The modular discriminant is $\Delta = g_2(\tau)^3 - 27g_3(\tau)^2$. The discriminant of an elliptic curve $y^2 = x^3 + Ax + B$ is $\Delta = -16(4A^3 + 27B^2)$ We also have the following lemma.

Theorem 2.7. *For any lattice Λ, the modular discriminant $\Delta(\tau)$ is always non-zero.*

Two prove this theorem, we first need two other lemmas, the first of which we state without proof.

Lemma 2.8. *Let $f(z)$ be an elliptic function for a lattice Λ. When counted with multiplicity, the number of zeros of $f(z)$ in any fundamental parallelogram F_α for Λ is equal to the number of poles of $f(z)$ in F_α.*

Lemma 2.9. *A point $z \notin L$ is a zero of $\wp'(z; L)$ if and only if $2z \in L$.*

Proof. Suppose $2z \in L$ for some $z \notin L$. Then

$$\wp'(z) = \wp'(z - 2z) = \wp'(-z) = -\wp'(z) = 0$$

where we have used the fact that $\wp'(z)$ is both periodic with respect to L and an odd function. If $L = [\omega_1, \omega_2]$, then

$$\frac{\omega_1}{2}, \quad \frac{\omega_2}{2}, \quad \frac{\omega_1 + \omega_2}{2}$$

are the only points $z \in \mathcal{F}_0$ that are not in L and also satisfy $2z \in L$. Since $\wp'(z)$ is an elliptic function of order 3 , it has only these three zeros in $\mathcal{F}_0$, by Lemma 2.9. Thus for any $z \notin L$ we have $\wp'(z) = 0$ if only if $2z \in L$. ■

Now we prove Theorem 2.8.

Proof of Theorem 2.8. Let $\Lambda = [\omega_1, \omega_2]$ and put

$$r_1 = \frac{\omega_1}{2}, \quad r_2 = \frac{\omega_2}{2}, \quad r_3 = \frac{\omega_1 + \omega_2}{2}.$$

Then $r_i \notin \Lambda$ and $2r_i \in \Lambda$ for $i = 1, 2, 3$. So $\wp'(r_i) = 0$ by Lemma 2.10. The differential equation for $\wp(z)$ corresponds to the curve $y^2 = 4x^3 - g_2(\Lambda) - g_3(\Lambda)$. From this, we see that $\wp(r_1), \wp(r_2)$, and $\wp(r_3)$ are the zeros of the cubic $f(x) = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$. Now the discriminant $\Delta(f)$ of $f(x)$ is equal to $16\Delta(\Lambda)$, thus

$$\Delta(L) = \frac{1}{16} \prod_{i<j} (\wp(r_i) - \wp(r_j))^2$$

and it suffices to show that the $\wp(r_i)$ are distinct. Let $g_i(z) = \wp(z) - \wp(r_i)$. Then $g_i(z)$ is an elliptic function of order 2 (its poles are the poles of $\wp(z)$), so it has exactly 2 zeros, by Lemma 2.9. Now r_i is a double zero because $g_i'(z) = \wp'(r_i) = 0$, by Lemma 2.10. Thus $g_i(z)$ has no other zeros, and therefore $\wp(r_j) \neq \wp(r_i)$ for $i \neq j$. ■

This modular discriminant is a modular form of weight 12 and g_2 is a modular form of weight 4. Cubing g_2, we get a modular form of weight 12. Thus j is a function of weight 0.

We also have the following lemma.

Theorem 2.10. *The j-invariant is holomorphic on $\mathbb{H}$.*

To prove this theorem, we need the following lemma.

Lemma 2.11. *For any lattice Λ, the sum $\sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^k}$ converges absolutely for all $k > 2$.*

Proof. Let δ be the minimum distance between points in Λ. Consider an annulus A of inner radius r and width $\frac{\delta}{2}$.

Any two distinct lattice points in A must be separated by an arc of length at least $\delta/2$ when measured along the inner rim of A. It follows that A contains at most $4\pi r/\delta$ lattice points. The number of lattice points in the annulus $\{\omega : n \leq |\omega| < n+1\}$ is therefore bounded by cn, where $c \leq (2/\delta)(4\pi r/\delta) = 8\pi/\delta^2$. We then have

$$\sum_{\omega \in \Lambda, |\omega| \geq 1} \frac{1}{|\omega|^k} \leq \sum_{n=1}^{\infty} \frac{cn}{n^k} = c \sum_{n=1}^{\infty} \frac{1}{n^{k-1}} < \infty,$$

since $k > 2$. The finite sum

$$\sum_{\omega \in \Lambda, 0|\omega| < 1} \frac{1}{|\omega|^k}$$

is bounded, thus

$$\sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{|\omega|^k} = \sum_{\omega \in \Lambda, 0 < |\omega| < 1} \frac{1}{|\omega|^k} + \sum_{\omega \in \Lambda, |\omega| \geq 1} \frac{1}{|\omega|^k} < \infty,$$

so the sum converges absolutely as claimed.

Note that this implies that g_2 and g_3 converge absolutely.

■

Now we will prove Theorem 2.11.

Proof of Theorem 2.11. By Lemma 2.12, g_2 and g_3 converge absolutely for any fixed $\tau \in \mathbb{H}$ and uniformly over τ in any compact subset of $\mathbb{H}$. The proof of the last fact is slightly technical. It follows that $g_2(\tau)$ and $g_3(\tau)$ are both holomorphic on $\mathbb{H}$, and therefore the modular discriminant $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$ is also holomorphic on $\mathbb{H}$. We know that $\Delta(\tau)$ is nonzero for all $\tau \in \mathbb{H}$. Thus the j-invariant $j(\tau)$ is holomorphic on $\mathbb{H}$ as well. ■

We also have the following property of the j-invariant:

Theorem 2.12. *Let L and L' be two lattices in $\mathbb{C}$. Then $j(L) = j(L')$ if and only if L and L' are homothetic.*

Proof. ■

Proof. First we prove the $\Leftarrow$ direction. Suppose L and L' are homothetic; that is, $L' = \lambda L$ for some $\lambda \in \mathbb{C}$. We know

$$g_2(L') = g_2(\lambda L) = 60 \sum_{\omega \in L \setminus \{0\}} \frac{1}{(\lambda\omega)^4} = \frac{1}{\lambda^4} 60 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4} = \lambda^{-4} g_2(L)$$

Similarly,

$$g_3(L') = g_3(\lambda L) = 140 \sum_{\omega \in L \setminus \{0\}} \frac{1}{(\lambda\omega)^6} = \frac{1}{\lambda^6} 140 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6} = \lambda^{-6} g_3(L).$$

So then

$$\begin{aligned} j(L') &= 1728 \cdot \tfrac{g_2(L')^3}{g_2(L')^3 - 27 g_3(L')^2} = 1728 \cdot \tfrac{\lambda^{-12} g_2(L)^3}{\lambda^{-12} g_2(L)^3 - 27\lambda^{-12} g_3(L)^2} \\ &= 1728 \cdot \tfrac{g_2(L)^3}{g_2(L)^3 - 27 g_3(L)^2} = j(L). \end{aligned}$$

Now we prove the reverse implication. Let $j(L) = j(L')$. Suppose we can find a $\lambda \in \mathbb{C}$ such that $g_2(L') = \lambda^{-4} g_2(L)$ and $g_3(L') = \lambda^{-6} g_3(L)$. Then $g_2(L') = g_2(\lambda L)$ and $g_3(L) = g_3(\lambda L)$, and by Lemma 2.2.6, we have that the Laurent expansion for $\wp(z; L')$ is

$$\wp(z; L') = \frac{1}{z^2} + \sum_{n=1}^{\infty} p(g_2(L'), g_3(L')) z^{2n} = \frac{1}{z^2} + \sum_{n=1}^{\infty} p(g_2(\lambda L), g_3(\lambda L)) z^{2n} = \wp(z; \lambda L)$$

Thus $\wp(z; L')$ and $\wp(z; \lambda L)$ have the same Laurent expansion about 0. So then these two functions agree on a neighborhood U about the origin. But we have that $\wp(z; L')$ and $\wp(z; \lambda L)$ are analytic on the region $\Omega := \mathbb{C} \setminus (\lambda L \cup L')$, and the set

$$\{z \in \Omega : \wp(z; L') = \wp(z; \lambda L)\}$$

certainly has a limit point in $U \cap \Omega$, and hence

$$\wp(z; L') = \wp(z; \lambda L)$$

on all of Ω, so $\wp(z; L')$ and $\wp(z; \lambda L)$ must have the same poles. Since, by Lemma 2.2.5, the lattice L' is precisely the set of poles of $\wp(z; L')$, then $L' = \lambda L$. Thus, to complete this proof, we need only find a $\lambda \in \mathbb{C}$ such that $g_2(L') = \lambda^{-4} g_2(L)$ and $g_3(L') = \lambda^{-6} g_3(L)$. Note that

$$g_2(L)^3 - 27 g_3(L)^2 = \Delta(L)$$

can never be 0. Thus, we have the following three cases:

- Case 1: $g_2(L') = 0$ and $g_3(L') \neq 0$. Then choose λ so that $\lambda^6 = \frac{g_3(L)}{g_3(L')}$.
- Case 2: $g_3(L') = 0$ and $g_2(L') \neq 0$. Then choose λ so that $\lambda^4 = \frac{g_2(L)}{g_2(L')}$.
- Case 3: $g_2(L') \neq 0$ and $g_3(L') \neq 0$. Then choose λ such that $\lambda^4 = \frac{g_2(L)}{g_2(L')}$. Since we have that $j(L) = j(L')$, then

$$1728 \cdot \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2} = 1728 \cdot \frac{g_2(L')^3}{g_2(L')^3 - 27g_3(L')^2}$$

By substituting $\lambda^4 g_2(L')$ for $g_2(L)$, we get that

$$\frac{(\lambda^4 g_2(L'))^3}{(\lambda^4 g_2(L'))^3 - 27g_3(L)^2} = \frac{g_2(L')^3}{g_2(L')^3 - 27g_3(L')^2}$$

Cross-multiplying and solving for λ^{12} yields that

$$\lambda^{12} = \frac{-27g_3(L)^2}{-27g_3(L')^2} = \frac{g_3(L)^2}{g_3(L')^2}$$

So then

$$\lambda^6 = \pm\frac{g_3(L)}{g_3(L')}$$

Assume the sign on the right is $+$, since if not, we can replace λ by $i\lambda$. Thus we have that $g_3(L') = \lambda^{-6} g_3(L)$ Hence, in any of the cases, we can find a $\lambda \in \mathbb{C}$ such that $g_2(L') = \lambda^{-4} g_2(L)$ and $g_3(L') = \lambda^{-6} g_3(L)$ ■

With this, we see that Lemma 2.6 follows.

Corollary 2.13. *$j(\tau + 1) = j(\tau)$.*

Proof. The lattices $[1, \tau]$ and $[1, \tau+1]$ are equal. Thus by Theorem 2.13, $j(\tau) = j(\tau+1)$. ■

3. Properties of the j-invariant

We know that $\Gamma = SL_2(\mathbb{Z})$ acts on H via the linear fractal transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d},$$

which implies that the $j-$invariant is invariant under the action of the modular group. In fact, we also have the following:

Lemma 3.1. *We have $j(\tau) = j(\tau')$ if and only if $\tau' = \gamma\tau$ for some gamma $\in \Gamma$.*

Proof. We have $j(S\tau) = j(-1/\tau) = j(\tau)$ and $j(T\tau) = j(\tau + 1) = j(\tau)$, by Theorem 2.13, It follows that if $\tau' = \gamma\tau$ then $j(\tau') = j(\tau)$, since S and T generate Γ.

To prove the converse, let us suppose that $j(\tau) = j(\tau')$. Then by Corollary 2.14, the lattices $[1, \tau]$ and $[1, \tau']$ are homothetic So $[1, \tau'] = \lambda[1, \tau]$, for some $\lambda \in \mathbb{C}^\times$. There thus exist integers a, b, c, and d such that

$$\tau' = a\lambda\tau + b\lambda$$
$$1 = c\lambda\tau + d\lambda$$

From the second equation, we see that $\lambda = \frac{1}{c\tau+d}$. Substituting this into the first, we have

$$\tau' = \frac{a\tau + b}{c\tau + d} = \gamma\tau, \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2}.$$

Similarly, using $[1, \tau] = \lambda^{-1}[1, \tau']$, we can write $\tau = \gamma'\tau'$ for some integer matrix γ'. The fact that $\tau' = \gamma\gamma'\tau'$ implies that $\det \gamma = \pm 1$ (since γ and γ' are integer matrices). But τ and τ' both lie in $\mathbb{H}$, so we must have $\det \gamma = 1$; therefore $\gamma \in \Gamma$ as desired. ■

Lemma 3.2. *We have that*

$$\lim_{\mathrm{im}(\tau)\to\infty} j(\tau) = \infty.$$

Proof. To start, consider

$$g_2(\tau) = 60 \sum_{m,n} \frac{1}{(m+n\tau)^4} = 60 \left(2 \sum_{m=1}^{\infty} \frac{1}{m^4} + \sum_{m,n;n\neq 0} \frac{1}{(m+n\tau)^4} \right) \tag{3.1}$$

Let $\tau = a + bi$. But if we consider a single term in the right-hand sum, we find that

$$\lim_{\mathrm{Im}(\tau)\to\infty} \frac{1}{(m+n\tau)^4} = \lim_{\mathrm{Im}(\tau)\to\infty} \frac{1}{m^4 + 4m^3(n\tau) + 6m^2(n\tau)^2 + 4m(n\tau)^3 + (n\tau)^4}$$

Because $n \neq 0$, then the b^4 term in $(n\tau)^4 = n^2\left(a^4 + 4a^3(bi) - 6a^2b^2 + 4a(bi)^3 - b^4\right)$ dominates the denominator as b becomes arbitrarily large. Hence,

$$\lim_{\mathrm{Im}(\tau)\to\infty} \frac{1}{(m+n\tau)^4} = 0$$

Since $g_2(\tau)$ is uniformly convergent by Lemma 2.8, then, in taking the limit, equation (3.1) becomes

$$\lim_{\mathrm{Im}(\tau)\to\infty} g_2(\tau) = 120 \sum_{m=1}^{\infty} \frac{1}{m^4}$$

But this is a known infinite sum, with $\sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{90}$. Hence.

$$\lim_{\mathrm{Im}(\tau)\to\infty} g_2(\tau) = \frac{4}{3}\pi^4$$

The limit behavior of $g_3(\tau)$ is shown similarly, with the sum $\sum_{m=1}^{\infty} \frac{1}{m^6} = \frac{\pi^6}{945}$ giving us that

$$\lim_{\mathrm{Im}(\tau)\to\infty} g_3(\tau) = \frac{8}{27}\pi^6$$

Combining these two results, we get the limit of the denominator of the j -function:

$$\lim_{\mathrm{Im}(\tau)\to\infty} \left[g_2(\tau)^3 - 27g_3(\tau)^2\right] = \left(\frac{4}{3}\pi^4\right)^3 - 27\left(\frac{8}{27}\pi^6\right)^2 = 0 \tag{3.2}$$

Thus, it follows that

$$\lim_{\infty\to\infty} j(\tau) = \infty.$$

■

Theorem 3.3. *The j-invariant is surjective.*

Proof. We know that $j(\tau)$ is nonconstant on $\mathbb{H}$, since there are values for τ that are distinct under the action of $\mathrm{SL}_2(\mathbb{Z})$ (each point in the fundamental region F is $\mathrm{SL}_2(\mathbb{Z})$ -distinct from every other point in F). Also, by Lemma 2.9, $j(\tau)$ is holomorphic on $\mathbb{H}$. Hence, by the open mapping theorem, the image of $j(\tau)$ must be an open set in $\mathbb{C}$. Since the only set that is both open and closed in $\mathbb{C}$ is itself, it is sufficient to prove that $j(\mathbb{H})$ is closed.

Let $j(\tau_k)$ be a sequence in $j(\mathbb{H})$ converging to some $w \in \mathbb{C}$ (where $\tau_k \in \mathbb{H}$). Since $j(\tau)$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$, then by only considering τ in our fundamental domain F, we may assume that each τ_k is such that

$$|\mathrm{Re}(\tau_k)| \leq \frac{1}{2} \text{ and } |\mathrm{Im}(\tau_k)| \geq \frac{\sqrt{3}}{2}\ .$$

Suppose the imaginary parts of the τ_k 's are unbounded. But then, by Lemma 3.2, $j(\tau_k)$ contains a subsequence converging to ∞. Since $j(\tau_k)$ converges to w, this cannot happen. So the imaginary parts of the τ_k 's are bounded, say by some $M \in \mathbb{R}$. Hence each τ_k lies in the region

$$R = \left\{ \tau \in \mathbb{H} : |\mathrm{Re}(\tau)| \leq \frac{1}{2}, \frac{\sqrt{3}}{2} \leq |\mathrm{Im}(\tau)| \leq \mathrm{M} \right\}$$

a compact subspace of $\mathbb{H}$. But this implies that τ_k has a subsequence converging to some $\tau_0 \in \mathbb{H}$. Since $j(\tau)$ is continuous and $j(\tau_k)$ converges to w, then $j(\tau_0) = w$, and hence $w \in j(\mathbb{H})$. Thus $j(\mathbb{H})$ is closed, and so $j(\mathbb{H}) = \mathbb{C}$.

■

We will state the next theorem without proof:

Theorem 3.4 (Schneider (1937)). *Suppose $\tau \in \mathbb{H}$ is algebraic and not an imaginary quadratic, then the value of $j(\tau)$ is transcendental.*

q-EXPANSION

We begin by defining the map $q : \mathbb{H} \to \mathbb{D}$ as

$$q(\tau) = e^{2\pi i \tau}.$$

We can write this as $e^{-2\pi i \tau}(\cos(2\pi\,\mathrm{re}(\tau)) + i \sin(2\pi\,\mathrm{re}(\tau)))$. We can see that map bijectively maps each vertical strip $\mathbb{H}_n := \{\tau \in \mathbb{H} : n \leq \mathrm{re}\ \tau < n+1\}$(for any $n \in \mathbb{Z}$) to the punctured unit disk $\mathbb{D}_0 := \mathbb{D} - \{0\}$. Also note that

$$\lim_{\tau \to \infty} q(\tau) = 0.$$

If $f : \mathbb{H} \to \mathbb{C}$ is a meromorphic function that satisfies $f(\tau+1) = f(\tau)$ for all $\tau \in \mathbb{H}$, then we can write f in the form $f(\tau) = f^*(q(\tau))$, where $f^* : \mathbb{D}_0 \to \mathbb{C}$ is a meromorphic function that we can define by fixing a vertical strip $\mathbb{H}_n$ and putting $f^* := f \circ \left(q_{|\mathbb{H}_n}\right)^{-1}$.

The q -expansion of $f(\tau)$ is obtained by composing the Laurent-series expansion of f^* at 0 with the function $q(\tau)$

$$f(\tau) = f^*(q(\tau)) = \sum_{n=-\infty}^{+\infty} a_n q(\tau)^n = \sum_{n=-\infty}^{+\infty} a_n q^n.$$

We typically just write q for $q(\tau)$.

We have the following the following lemma:

Lemma 3.5. *Let $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$ and let*

$$\sigma_\ell(n) = \sum_{d|n} d^\ell$$

be the sum of the ℓ th powers of the positive divisors of n. If $k \geq 2$ is an integer, then

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$$
$$= 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{j=1}^{\infty}\frac{j^{2k-1}q^j}{1-q^j}$$

Proof. We have

$$\pi\frac{\cos\pi\tau}{\sin\pi\tau} = \pi i\frac{e^{\pi i\tau}+e^{-\pi i\tau}}{e^{\pi i\tau}-e^{-\pi i\tau}}$$
$$= \pi i\frac{q+1}{q-1} = \pi i + \frac{2\pi i}{q-1}$$
$$= \pi i - 2\pi i\sum_{j=0}^{\infty}q^j$$

Using the product expansion,

$$\sin\pi\tau = \pi\tau\prod_{n=1}^{\infty}\left(1-\frac{\tau}{n}\right)\left(1+\frac{\tau}{n}\right)$$

and taking the logarithmic derivative yields

$$\pi\frac{\cos\pi\tau}{\sin\pi\tau} = \frac{1}{\tau} + \sum_{n=1}^{\infty}\left(\frac{1}{\tau-n}+\frac{1}{\tau+n}\right).$$

Differentiating the first equation and the above equation $2k-1$ times with respect to τ yields

$$-\sum_{j=1}^{\infty}(2\pi i)^{2k}j^{2k-1}q^j = (-1)^{2k-1}(2k-1)!\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^{2k}}.$$

Consider the G_k expansion with $2k$ in place of k. Since $2k$ is even, the terms for (m,n) and $(-m,-n)$ are equal, so we only need to sum for $m=0, n>0$ and for $m>0, n\in\mathbf{Z}$, then double the answer. We obtain

$$G_{2k}(\tau) = 2\sum_{n=1}^{\infty}\frac{1}{n^{2k}} + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(m\tau+n)^{2k}}$$
$$= 2\zeta(2k) + 2\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{(2\pi i)^{2k}j^{2k-1}}{(2k-1)!}q^{mj}$$
$$= 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}j^{2k-1}q^{mj}$$

Let $n = mj$ in the last expression. Then, for a given n, the sum over j can be regarded as the sum over the positive divisors of n. This yields the first expression in the statement of the proposition. The expansion $\sum_{m\geq 1}q^{mj} = q^j/(1-q^j)$ yields the second expression. ∎

Using the facts that

$$\zeta(4) = \frac{\pi^4}{90} \quad\text{and}\quad \zeta(6) = \frac{\pi^6}{945},$$

we obtain

$$g_2(\tau) = \tfrac{4\pi^4}{3}(1+240q+\cdots) = \tfrac{4\pi^4}{3}\left(1+240\sum_{j=1}^{\infty}\tfrac{j^3q^j}{1-q^j}\right)$$
$$g_3(\tau) = \tfrac{8\pi^6}{27}(1-504q+\cdots) = \tfrac{8\pi^6}{27}\left(1-504\sum_{j=1}^{\infty}\tfrac{j^5q^j}{1-q^j}\right)$$

Since $\Delta = g_2^3 - 27g_3^2$, a straightforward calculation shows that

$$\Delta(\tau) = (2\pi)^{12}(q+\cdots)$$

Then $j(\tau) = \frac{1}{q} + \cdots$. Including a few more terms in the above calculations yields

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots.$$

The coefficents of the expansions are always integers, and this results is several almost integers, in particular, the Ramanujan constant, $e^{\pi\sqrt{163}} = 262537412640768743.9999999999992$. The details of this, require results from Complex Multiplication, and thus, we will omit them. However, we give a brief explanation of why this is true using q-expansions:

We first state the following theorem:

Theorem 3.6. *For algebraic numbers α and β with $\alpha \neq 0$ and $\beta \notin \mathbb{Q}$ and for any choice of $\log(\alpha) \neq 0$, the number α^β is transcendental.*

Note that this implies that the number $e^{\pi\sqrt{163}}$ is transcendental and we may use Theorem 3.4. Now let $\tau = \frac{+i\sqrt{163}}{2}$. Then $q = -e^{-\pi\sqrt{163}}$. Then we have

$$j(\tau) = -(640320)^3$$

and

$$\left|j(\tau) - \frac{1}{q} - 744\right| < 10^{-12}.$$

Therefore

$$\left|-(640320)^3 - 744 + e^{\pi\sqrt{163}}\right| < 10^{-12}$$

and thus $j(\tau)$ is a good approximation of $e^{\pi\sqrt{163}}$.

More on Modular Functions

We first give the definition of a *modular function.*

Definition 3.7. A function $f : \mathbb{H} \to \mathbb{C}$ is called a modular function if it satisfies the following:

1. f is meromorphic on $\mathbb{H}$.
2. $f(\gamma \cdot \tau) = \tau$ for any $\gamma \in SL_2(\mathbb{Z})$ and all $\tau \in \mathbb{H}$.
3. The function f has a q- expansion at ∞ of the form

$$f(\tau) = \sum_{n=-m}^{\infty} c(n)q^n$$

where $q = e^{2\pi i\tau}$, and m is some integer.

Lemma 3.8. *The j-invariant is a modular function for $SL_2(\mathbb{Z})$.*

Proof. We know from Equation (3.2) in Lemma 3.2 that $\Delta(\tau)$ has a simple zero at ∞, then $j(\tau)$ has a simple pole at ∞. This implies that $j(\tau)$ is meromorphic at ∞. Combining this with Theorem 3.1, we see that the lemma must hold. ■

We also have the following.

Lemma 3.9. *A modular function f which is holomorphic at ∞, that is, its q-expansion has no negative powers of q, is constant.*

Proof. It suffices to show that $f(\mathbb{H}\cup\{\infty\})$ is compact in $\mathbb{C}$; the maximum modulus principle of complex analysis then tells us that f must be constant. We do so by showing that $f(\mathbb{H}\cup\{\infty\})$ is sequentially compact, i.e., every sequence has a subsequence that converges to a point in $f(\mathbb{H}\cup\{\infty\})$. Since $\mathbb{C}$ is a metric space, compactness is equivalent to sequential compactness.

Let $\{f(\tau_k)\}$ be a sequence in $f(\mathbb{H}\cup\{\infty\})$. Since f is modular and thus $\mathrm{SL}_2(\mathbb{Z})$ -invariant, we can assume that each τ_k lies in our fundamental region F. If the imaginary parts of the τ_k 's are unbounded, then there is a subsequence of $\{\tau_k\}$ converging to $i\infty$. But then there is a subsequence of $\{f(\tau_k)\}$ which converges to $f(\infty)$, which is a finite complex number since f is holomorphic at ∞.

If the imaginary parts of the τ_k 's are bounded, say $\mathrm{Im}(\tau_k)\leq M$ for some $M\in\mathbb{R}$, then each τ_k lies in the region $R:=\{z\in\mathbb{C}:|z|\geq 1,-1/2\leq \mathrm{Re}(z)\leq 1/2, \mathrm{Im}(z)\leq M\}$, which is closed and bounded and hence compact. Thus a subsequence of $\{\tau_k\}$ can be found which converges to an element τ_0 of R, and hence $f(\tau_k)$ converges to $f(\tau_0)$, since f is continuous. Hence, $f(\mathbb{H}\cup\{\infty\})$ is compact and thus f must be constant. ■

Lemma 3.10. *Every holomorphic modular function for $SL_2(\mathbb{Z})$ is a polynomial in $j(\tau)$.*

Proof. Suppose $f(\tau)$ is a holomorphic modular function for $\mathrm{SL}_2(\mathbb{Z})$. Since f is modular and thus meromorphic at the cusp $i\infty$, its q -expansion looks like

$$f(\tau)=\sum_{n=-m}^{\infty} a_n q^n$$

where m is some positive integer. But the only negative q -power term in $j(\tau)$'s q -expansion is simply q^{-1}. Hence, we can define a polynomial $A(x)$ such that $f(\tau)-A(j(\tau))$ has no terms with negative q powers. So then $f(\tau)-A(j(\tau))$ is holomorphic at ∞ and hence must be constant by Lemma 3.7. Thus,

$$f(\tau)=k+A(j(\tau))$$

for some $k\in\mathbb{C}$ and hence $f(\tau)$ is a polynomial in $j(\tau)$. ■

Theorem 3.11. *Every modular function for $SL_2(\mathbb{Z})$ is a rational function in $j(\tau)$.*

Proof. Suppose $f(\tau)$ is a modular function for $\mathrm{SL}_2(\mathbb{Z})$. Our goal is to find some polynomial $B(x)$ such that $B(j(\tau))f(\tau)$ is holomorphic. Then by Lemma 3.8, we are done.

Since f is modular, it only has a finite number of poles in our fundamental domain F. The idea is to find a polynomial of $j(\tau)$ for each pole that kills each pole of $f(\tau)$, making the resulting function holomorphic at that point. Let τ_0 be a pole of f of order m. Suppose $j'(\tau_0)\neq 0$. Then consider

$$(j(\tau)-j(\tau_0))^m f(\tau)$$

If we write $j(\tau)$ and $f(\tau)$ as Laurent series about τ_0, we get

$$j(\tau)=\sum_{n=0}^{\infty} a_n\left(\tau-\tau_0\right)^n \text{ and } f(\tau)=\sum_{n=-m}^{\infty} b_n\left(\tau-\tau_0\right)^n$$

for some constants a_n and b_n since, at the point τ_0, we know j is holomorphic and f has a pole of order m. Note that $a_0=j\left(\tau_0\right)$. Then consider

$$\begin{aligned}
\left(j(\tau)-j\left(\tau_0\right)\right)^m f(\tau) &= \left(\left(\sum_{n=0}^{\infty} a_n\left(\tau-\tau_0\right)^n\right)-a_0\right)^m f(\tau) \\
&= \left(\sum_{n=1}^{\infty} a_n\left(\tau-\tau_0\right)^n\right)^m \sum_{n=-m}^{\infty} b_n\left(\tau-\tau_0\right)^n .
\end{aligned}$$

The resulting expansion will have no negative powers of $(\tau-\tau_0)$, and hence

$$\left(j(\tau)-j\left(\tau_0\right)\right)^m f(\tau)$$

is holomorphic at τ_0. Multiplying all such polynomials $\left(j(\tau)-j\left(\tau_k\right)\right)^{m_k}$ corresponding to each of the k poles of f in F yields the polynomial in $j(\tau)$

$$\prod_k\left(j(\tau)-j\left(\tau_k\right)\right)^{m_k}$$

which, when multiplied by f, gives a function that is holomorphic at each pole τ_k. The only case we have omitted is if we have a pole τ_0 such that $j'\left(\tau_0\right)=0$. It turns out that this only happens at i and $e^{2\pi i/3}$, and similar polynomials in $j(\tau)$ are easily obtained at each point so that their product with $f(\tau)$ gives a suitably holomorphic function. Thus, multiplying all of our collected polynomials together to get some $B(j(\tau))$, we have that $B(j(\tau))f(\tau)$ is holomorphic, and hence by our previous theorem.

$$f(\tau)=\frac{A(j(\tau))}{B(j(\tau))}$$

for some polynomial $A(x)$. ■

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