Quaternionic Analysis and Special Relativity

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May 2020

1 Introduction

In this paper, we study quaternions in depth in the context of both algebra and analysis. Quaternions were initially introduced to the world of mathematics, by William Rowan Hamilton in 1843. Though they are not studied quite as often, their usefulness extends beyond just math. This paper will also explore the extension of quaternions to the theory of special relativity. We will start by establishing basic quaternion arithmetic. We will then introduce unit quaternions and interesting properties regarding rotations. The article [5] provides foundation for the information on both quaternion arithmetic and rotations. After these basic notions are established, we then proceed with developing a quaternion definition of regular functions, and the properties that make a function regular ([4] provides further detail into regular functions for quaternions). This allows us state and prove the Cauchy-Riemann-Fueter Equations, an analog of the Cauchy-Riemann equations in complex analysis. We move forward to prove two elemental proofs in analysis: Cauchy's Theorem and his Integral Formula. Much of the Cauchy generalizations for quaternions are thanks to the Swiss mathematician Fueter. However, two papers, one by A. Sudbery and another by C. Deavours [7] [3], provide very detailed explanations of such topics. Finally, we move into a brief discussion of special relativity and the motivation for using quaternions to derive the theory. More information can be found in the papers of V. Ariel [1] [2].

2 Quaternion Arithmetic

Before defining the quaternion itself, we start with looking at the Q_8 quaternion group. The quaternion group is a non-abelian group with exactly eight elements: $\{1, -1, i, -i, j, -j, k, -k\}$. The elements of the group behave in the following manner:

$$i^2 = j^2 = k^2 = -1,\; ij = k = -ji,\; jk - i = -kj,\; ki = j = -ik$$

With that being established, we can proceed with defining quaternions:

Definition 2.1. *A Quaternion is a number of the form* $q = w + xi + yj + zk$ *where* $w, x, y, z \in \mathbb{R}$ *and* $i, j,$ *and* k *are imaginary numbers such such that* $i^2 = j^2 = k^2 = ijk = -1$.

Addition and subtraction are termwise, and so is multiplication but the latter is not commutative.

Definition 2.2. *The Conjugate of a quaternion is given by* $q^* = w - xi - yj - zk$.

The conjugate of a product of quaternions is the product of their respective conjugates, so we have $(pq)^* = p^*q^*$. Furthermore, $(p^*)^* = p$.

Definition 2.3. *The Norm of a quaternion is given by* $N(q) = w^2 + x^2 + y^2 + z^2$.

It is a real-valued function where $N(p)N(q) = N(pq)$ and $N(q^*) = N(q)$.

Definition 2.4. *The multiplicative inverse of a quaternion is given by* $q^{-1} = \frac{q^*}{N(q)}$ *where division of a quaternion by a scalar is done componentwise.*

Unsurprisingly, the inverse satisfies $(p^{-1})^{-1} = p$ and $(pq)^{-1} = q^{-1}p^{-1}$.

Next, we want a function that can select the real part of a quaternion; we'll denote it as $W(q)$ and call it the selection function. It is possible to split a quaternion into its real part an a 3D vector by writing it like $q = w + \hat{v}$ where $\hat{v} = xi + yj + zk$. It can also be written as (x, y, z).

Now, we can represent multiplication through this form, using dot products and cross products. We have

$$(w_0 + \hat{v}_0)(w_1 + \hat{v}_1) = (w_0w_1 - \hat{v}_0 \cdot \hat{v}_1) + w_0\hat{v}_1 + w_1\hat{v}_0 + \hat{v}_0 \times \hat{v}_1$$

Using this form, we can now understand when multiplication is commutative. We see that it happens when the two 3D vectors are parallel. Thus, $q_0q_1 = q_1q_0$ if and only if $\hat{v}_0 \times \hat{v}_1 = 0$.

Furthermore, as their name suggests, quaternions can be represented as 4D vectors of the form (w, x, y, z). Let's define their dot product.

Definition 2.5. *The dot product of two quaternions is*

$$q_0 \cdot q_1 = w_0w_1 + x_0x_1 + y_0y_1 + z_0z_1 = W(q_0q_1^*)$$

3 Unit Quaternions

Definition 3.1. *A unit quaternion is a quaternion* q *for which* $N(q) = 1$.

The inverse of a unit quaternion and the product of unit quaternions are also unit quaternions. We can also define them trigonometrically as such:

$$q = \cos\theta + \hat{u}\sin\theta$$

where $\hat{u}$ is a 3D vector with length one. If we square it, we get -1.

With complex numbers, we have Euler's Identity, and this can be generalized to quaternions, if we use 3D vectors to represent the imaginary components. We have

$$e^{(\hat{u}\theta)} = \cos\theta + \hat{u}\sin\theta$$

where the left-hand side is derived from substituting $\hat{u}\theta$ into the power series representation for e^x (and using the fact that $\hat{u}^2 = -1$. This representation can allow us to raise a quaternion to any power as such:

$$q^t = (\cos\theta + \hat{u}\sin\theta)^t = e^{(\hat{u}t\theta)} = \cos(t\theta) + \hat{u}\sin(t\theta)$$

Just as how we can define the exponential of a quaternion, we can define its logarithm. We have that

$$\log(q) = \log(\cos\theta + \hat{u}\sin\theta) = \log(e^{(\hat{u}\theta)}) = \hat{u}\theta$$

However, these are not quite as nice as we'd want them to be due to noncommutativity. Some of the implications are that familiar identities are not true fro the quaternions. For example, $e^p e^q$ may not be e^{p+q} and $\log(pq)$ and $\log(p)+\log(q)$ are not necessarily equal.

Let us now explore what happens when we try to rotate a quaternion. We want to perform an operation on a vector in $\mathbb{R}^3$, so we use unit quaternions. Recall that a unit quaternion can be written as $\cos\theta + \hat{u}\sin\theta$. Thus, we can define the operator on a vector $\boldsymbol{v} \in \mathbb{R}^3$ as such:

$$\begin{aligned} L_q(\boldsymbol{v}) &= q\boldsymbol{v}q^* \\ &= \left(q_0^2 - \|\boldsymbol{q}\|^2\right)\boldsymbol{v} + 2(\boldsymbol{q}\cdot\boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q}\times\boldsymbol{v}). \end{aligned}$$

This operator is length preserving, since

$$\begin{aligned} \|L_q(\boldsymbol{v})\| &= \|q\boldsymbol{v}q^*\| \\ &= |q|\cdot\|\boldsymbol{v}\|\cdot|q^*| \\ &= \|\boldsymbol{v}\| \end{aligned}$$

It also preserves the direction of $\boldsymbol{v}$. To show this, we let $\boldsymbol{v} = k\boldsymbol{q}$ and have

$$\begin{aligned} q\boldsymbol{v}q^* &= q(k\boldsymbol{q})q^* \\ &= \left(q_0^2 - \|\boldsymbol{q}\|^2\right)(k\boldsymbol{q}) + 2(\boldsymbol{q}\cdot k\boldsymbol{q})\boldsymbol{q} + 2q_0(\boldsymbol{q}\times k\boldsymbol{q}) \\ &= k\left(q_0^2 + \|\boldsymbol{q}\|^2\right)\boldsymbol{q} \\ &= k\boldsymbol{q} \end{aligned}$$

These properties make L_q a pretty good candidate for a rotation function about $\boldsymbol{q}$, and we will prove this soon.

Before proceeding with the theorem, we note that L_q is linear over $\mathbb{R}^3$ because for any two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3$ and any $a_1, a_2 \in R$ we can easily show that

$$L_q\left(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2\right) = a_1 L_q\left(\boldsymbol{v}_1\right) + a_2 L_q\left(\boldsymbol{v}_2\right).$$

Now we move on to the theorem.

Theorem 3.1. *For any unit quaternion*

$$q = q_0 + \boldsymbol{q} = \cos\frac{\theta}{2} + \boldsymbol{u}\sin\frac{\theta}{2}$$

and for any vector $\boldsymbol{v} \in \mathbb{R}^3$ *the action of the operator*

$$L_q(\boldsymbol{v}) = q\boldsymbol{v}q^*$$

on $\boldsymbol{v}$ *is equivalent to a rotation of the vector through an angle* θ *about* $\boldsymbol{u}$ *as the axis of rotation.*

Proof. Given a vector $\boldsymbol{v} \in \mathbb{R}^3$, we decompose it as $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{n}$, where $\boldsymbol{a}$ is the component along the vector $\boldsymbol{q}$ and $\boldsymbol{n}$ is the component normal to $\boldsymbol{q}$. We'll show that under the operator $L_q, \boldsymbol{a}$ is invariant, while n is rotated about q through an angle θ. Since the operator is linear, the image $q\boldsymbol{v}q^*$ can be represented as a rotation of $\boldsymbol{v}$ about $\boldsymbol{q}$ through an angle θ.

So we know that a is invariant under L_q. Let us now focus on the effect of L_q on the orthogonal component $\boldsymbol{n}$. We have

$$\begin{aligned} L_q(\boldsymbol{n}) &= \left(q_0^2 - \|\boldsymbol{q}\|^2\right)\boldsymbol{n} + 2(\boldsymbol{q}\cdot\boldsymbol{n})\boldsymbol{q} + 2q_0(\boldsymbol{q}\times\boldsymbol{n}) \\ &= \left(q_0^2 - \|\boldsymbol{q}\|^2\right)\boldsymbol{n} + 2q_0(\boldsymbol{q}\times\boldsymbol{n}) \\ &= \left(q_0^2 - \|\boldsymbol{q}\|^2\right)\boldsymbol{n} + 2q_0\|\boldsymbol{q}\|(\boldsymbol{u}\times\boldsymbol{n}) \end{aligned}$$

where in the last step above we introduced $\boldsymbol{u} = \boldsymbol{q}/\|\boldsymbol{q}\|$. Denote $\boldsymbol{n}_\perp = \boldsymbol{u}\times\boldsymbol{n}$. So the last equation becomes

$$L_q(\boldsymbol{n}) = \left(q_0^2 - \|\boldsymbol{q}\|^2\right)\boldsymbol{n} + 2q_0\|\boldsymbol{q}\|\boldsymbol{n}_\perp$$

Note that $\boldsymbol{n}_\perp$ and $\boldsymbol{n}$ have the same length:

$$\|\boldsymbol{n}_\perp\| = \|\boldsymbol{n}\times\boldsymbol{u}\| = \|\boldsymbol{n}\|\cdot\|\boldsymbol{u}\|\sin\frac{\pi}{2} = \|\boldsymbol{n}\|$$

Finally, we rewrite the equation into the form

$$\begin{aligned} L_q(\boldsymbol{n}) &= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\boldsymbol{n} + \left(2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)\boldsymbol{n}_\perp \\ &= \cos\theta\boldsymbol{n} + \sin\theta\boldsymbol{n}_\perp \end{aligned}$$

Namely, the resulting vector is a rotation of $\boldsymbol{n}$ through an angle θ in the plane defined by $\boldsymbol{n}$ and $\boldsymbol{n}_\perp$. This vector is clearly orthogonal to the rotation axis. □

Note that this method of representing rotations is more elegant than other representations, like orthogonal matrices, which contain 9 numbers. It is also easy to derive the quaternion that corresponds to a given axis and angle (and vice versa). Quaternions are often used in video game graphics, as they cleanly represent rotating points in 3D space.

4 Regular Quaternion Functions

In this section we want to develop the idea of an analytic function for quaternion variables. In order to proceed, the definitions of quaternion derivatives and the quaternion wedge product must be clearly established:

Definition 4.1. *We can regard the differential of a function $f : \mathbb{H} \to \mathbb{H}$ as a quaternion valued 1 form*

$$df = \frac{\partial f}{\partial w}dt + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Definition 4.2. *We consider a function $f : \mathbb{H} \to \mathbb{H}$ to be **left quaternion-differentiable** if the following limit exists at q:*

$$\lim_{h\to 0} \left[h^{-1}\{f(q+h) - f(q)\}\right]$$

Additionally, we define the wedge product in the usual way: if θ is an r -form and ϕ is an s -form,

$$\theta \wedge \phi\left(h_1, \ldots, h_{r+s}\right) = \frac{1}{r!s!}\sum_{\rho} \epsilon(\rho)\theta\left(h_{\rho(1)}, \ldots, h_{\rho(r)}\right)\phi\left(h_{\rho(r+1)}, \ldots, h_{\rho(r+s)}\right)$$

where the sum is over all permutations ρ of $r+s$ objects, and $\epsilon(\rho)$ is the sign of ρ.

Knowing these definitions, we can look in to the following theorem to get a better understanding of analytic functions in quaternions and their analogs to complex analytic functions.

Theorem 4.1. *Suppose for a connected open set U, we have a defined function f, differentiable on the left. Then f has the following form on U, for some $a, b \in \mathbb{H}$*

$$f(q) = a + qb$$

Proof. We can start by equating the coefficients of the general quaternion - dw, dx, dy, dz - providing us with the following:

$$\frac{df}{dq} = \frac{\partial f}{\partial w} = i\frac{\partial f}{\partial x} = j\frac{\partial f}{\partial y} = k\frac{\partial f}{\partial z} \tag{1}$$

Now, we can set $q = v + jt$ where $v = w + ix$ and $t = y + iz$, and allow $f(q) = g(v,t) + jh(v,t)$, where g and h are complex functions of two complex variable v and t. This will allow us to separate equation (1) in to two sets of complex equations:

$$\frac{\partial g}{\partial w} = -i\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} = i\frac{\partial h}{\partial z},$$

$$\frac{\partial h}{\partial w} = i\frac{\partial h}{\partial x} = \frac{\partial g}{\partial y} = i\frac{\partial g}{\partial z}$$

When we represent these as complex derivatives, we can get the following:

$$\frac{\partial g}{\partial v} = \frac{\partial h}{\partial t} = \frac{\partial h}{\partial v} = \frac{\partial g}{\partial t} = 0 \tag{2}$$

$$\frac{\partial g}{\partial v} = \frac{\partial h}{\partial t} \tag{3}$$

$$\frac{\partial h}{\partial v} = \frac{\partial g}{\partial t} \tag{4}$$

Equation 2 shows that both h and g are in fact a complex analytic functions of v and t. Therefore, by Hartog's Extension Theorem [6], g and h have continuous partial derivative of all orders. Hence, from equation (3), we have the following:

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial h}{\partial w}\right) = \frac{\partial}{\partial w}\left(\frac{\partial h}{\partial v}\right) = 0$$

Now, suppose that U is a convex set. This allows us to deduce that g is linear in $\bar{t}$, h is linear in v and h is linear in $\bar{v}$. This would provide us with the following:

$$g(v,t) = \alpha + \beta v + \gamma \bar{t} + \delta(v\bar{t})$$

$$h(v,t) = \epsilon + \zeta v + \eta t + \theta v t$$

where the Greek letters each represent a different complex constant. Referring back to equations (3) and (4) we can see the following relations between the constants:

$$\beta = \eta_1 \quad \zeta = -\gamma_1 \quad \delta = \theta = 0.$$

Thus

$$\begin{aligned} f &= g + jh = \alpha + j\epsilon + (v + jw](\beta - j\gamma) \\ &- a + qb \end{aligned}$$

where $a = \alpha + je$ and $b = \beta - j\gamma_i$ so f is of the stated form if U is convex. Now to meet the actual requirements of the theorem of an open set U, we can use connected chains of convex sets which overlap in pairs. If we compare the form of the function f on the overlaps, it can be seen that $f(q) = a + qb$ with the same constants a, b throughout U. □

Although this proof provide more insight into the properties of regular functions for quaternions, such funcitons in general still do not satisfy Cauchy's theorem in the form

$$\int dqf = 0$$

. Thus, at this point, we define regular functions for quaternions and move to develop the Cauchy Riemann equations for quaternions (called the Cauchy-Riemann-Fueter equations):

Definition 4.3. *A function $f : \mathbb{H} \to \mathrm{H}$ is left-regular at $q \in \mathbb{H}$ if it is real differentiable at q and there exists a a quaternion $f'_t(q)$ such that*

$$d(dq \wedge dqf) = -2Dqf'_\ell(q)$$

It is right-regular if there exists a quaternion $f^t_r(q)$ such that

$$d(fdq \wedge dq) = -2f'_r(q)Dq$$

Since the definitions of left and right regular functions are clearly equivalent, we can only consider left regular functions for the sake of definiteness, which we can call regular. Additionally, we will also consider the following and call it the derivative of f at q : $f'(q) = f'_\ell(q)$. This will be done through the Cauchy Riemann Fueter equations. However, before we can state and prove the equations, we must define the Gamma map for quaternions:

$$\Gamma_f(df) = \frac{\partial f}{\partial w} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{y} + k\frac{\partial f}{\partial z}$$

Additionally, the following relation exists for the Gamma map, and is necessary in proving the Cauchy-Riemann-Fueter equations. A proof/derivation of the equation can be found in [7].

$$\Gamma_\ell(\alpha) = \alpha(1) + i\alpha(i) + j\alpha(j) + k\alpha(k) \tag{5}$$

We can now state and prove the Caucy Riemann Fueter equations:

Theorem 4.2. *(the Cauchy-Riemann-Fueter equations) A real-differentiable function f is regular at q if and only if it satisfies the following:*

$$\Gamma_r(df_q) = 0$$

With both such facts established, we can continue to the proof of the theorem.

Proof. Suppose that f is regular at q. Then, from the definition of regular functions, we have the following relation:

$$dq \wedge dq \wedge df_q = -2Ddf'(q)$$

We now evaluate these trilinear functions using two different set of arguments. The first is when we allow our arguments to be i, j, k and the second, is when we allow our arguments to be $1, i, j$:

$$(ij - ji)df_q(k) + (jk - kj)df_q(i) + (ki - ik)df_q(j) = -2f'(q)$$

$$(ij - ji)df_q(1) = 2kf'(q)$$

Comparing these two equations, we get the following final relationship:

$$f'(q) - df_0(1) - \quad \{idf_q(i) + jdf_q(j) + kdf_q(k)\} \tag{6}$$

If we compare equations (5) and (6), we can see that $\Gamma_r(df_q) = 0$.
When looking at the converse, we can see that if $\Gamma_r(df_q) = 0$, we can define $f\prime(q) - df_q(1)$. Evaluating this as we did above, will provide us with the following relation: $dq \wedge dq \wedge df_q = 2Ddf'(q)$. Hence, the function f is regular at q. □

5 Quaternion Analysis

In this section we will establish the necessary terms and theorems, before stating and proving both Cauchy's Theorem and the Integral Formula in the context of quaternions.To conduct this proof, we will start by proving these theorems for parallelepipeds. To proceed, it is important that we establish the following definition:

Definition 5.1. *An **oriented** k **-parallelepiped** in $\mathbb{H}$ is a map $C : I^k \to \mathbb{H}$, where $I^k \subset \mathbb{R}^k$ is the closed unit k cube, of the form*

$$C(t_1, \ldots, t_k) = q_0 + t_1 h_1 + \ldots + t_k h_k$$

*Now, $q_0 \in \mathbb{H}$ is called the **original vertex** of the parallelepiped, and $h_1, \ldots, h_k \in H$ are called its **edge-vectors**. A k -parallelepiped is non-degenerate if its edge-vectors are linearly independent (over $\mathbb{R}$) . A non - degenerate 4 -parallelepiped is positively oriented if $v(h_1, \ldots, h_4) > 0$, negatively oriented if $v(h_1, \ldots, h_4) < 0$.*

Now, in order to study Cauchy's theorem, it is important that we establish the following foundation:

Theorem 5.1. *A function f that is differentiable, is regular at q if and only if the following is satisfied:*

$$Dq \wedge df_q = 0$$

Proof.

$$\begin{aligned} Dq \wedge df_q(i,j,k,l) &= Dq(i,j,k)df_q(l) \quad Dq(j,k,l)df_q(i) \\ &+Dq(k,l,i)df_q(j) \quad Dq(l,i,j)df_q(k) \\ &= df_q(l) + idf_q(i) + jdf_q(j) + kdf_q(k) \\ &= \Gamma_r(df_q) \end{aligned}$$

Referring back to the Cauchy-Riemann-Fueter equations (Theorem 3.2), we can see that the above vanishes if and only if f is regular at q. □

With this foundation established, we can move toward stating and proving Cauchy's theorem and the integral formula for parallelepipeds.

Theorem 5.2 (Cauchy's theorem for a parallelepiped)**.** *If f is regular at every point of the 4-parallelepiped C,*

$$\int_{\partial C} Dqf = 0$$

Proof. Suppose q_0 and $h_1, \ldots h_4$ are the original vertex and edge vectors of C respectively. Then, for each subset S of $\{1,2,3,4\}$, allow C_s to be the 4-parallelepiped with edge-vectors $\frac{1}{2}h_1, \ldots, \frac{1}{2}h_4$ and original vertex $\sum_{i \in S} \frac{1}{2} H_i$. From this we can see the following:

$$\int_{\partial C_1} Dqf = \sum_S \int_{\partial C_S} Dqf$$

This means that there must exists some C_s, we will call it C_1, such that is statisfies the following:

$$\left|\int_{\partial C_1} Dqf\right| \geq \frac{1}{16}\left|\int_{\partial C} Dqf\right|$$

If we continue to further dissect C_1 in the same manner, we get a sequence pf 4-pareallelepiped C_n with $C \supset C_1 \supset C_2 \supset C_3 \supset \ldots$ and

$$\left|\int_{\partial C_n} Dqf\right| \geq \frac{1}{16^n}\left|\int_{\partial C} Dqf\right| \tag{7}$$

There must be a point $q_\infty \in \cap C_n$, and $q_n \to q_\infty$ as $n \to \infty$. Since f is real-differentiable at q_∞, we can write

$$f(q) = f\left(q_\infty\right) + \alpha\left(q - q_\infty\right) + \left(q - q_\infty\right) r(q)$$

where $\alpha = df_{q_\infty} \in F_1$, and $r(q) \to 0$ as $q \to q_\infty$. Then if we define $r\left(q_\infty\right) = 0$, r is a continuous function and so $|r(q)|$ has a maximum value ρ_n on ∂C_n. Since the C_n converge on $q_\infty, \rho_n \to 0$ as $n \to \infty$. Now

$$\int_{\partial C_n} Dqf(q) = \int_{C_n} d(Dq) f\left(q_\infty\right) = 0$$

and

$$\int_{\partial C_n} Dq\alpha\left(q - q_\infty\right) = \int_{C_n} d(Dq\alpha) = 16^n (Dq \wedge \alpha)\left(h_1, \ldots, h_4\right) = 0$$

by theorem 5.1 , since f is regular at q_∞. Thus

$$\int_{\partial C_n} Dqf(q) = \int_{\partial C_n} Dq\left(q - q_\infty\right) r(q)$$

Allow $F : I^3 \to \mathbb{H}$ to be one of the 3 -parallelepipeds that forms the faces C_n. Then $F \subset \partial C_n$, and the edge-vectors of F are three of the four edge-vectors of C_n, $2^{-n} h_a, 2^{-n} h_b$, and $2^{-n} h_c$. For $q \in F\left(I^3\right)$ we have $|r(q)| < \rho_n$ and $\left|q - q_\infty\right| \leq 2^{-n}\left(\left|h_1\right| + \ldots + \left|h_4\right|\right)$; hence

$$\left|\int_F Dq\left(q - q_\infty\right) r(q)\right| \leq 8^{-n}\left|Dq\left(h_a, h_b, h_c\right)\right| 2^{-n}\left(\left|h_1\right| + \ldots + \left|h_4\right|\right) \rho_n$$

Suppose V is the largest $\left|Dq\left(h_a, h_b, h_c\right)\right|$ for all of a, b, c; since the integral over ∂C_n is the sum of 8 integrals over faces F,

$$\left|\int_{\partial C_n} Dqf(q)\right| \leq 8.16^{-n} V\left(\left|h_1\right| + \ldots + \left|h_4\right|\right) \rho_n .$$

If we combine this with (7), we can see that

$$\left|\int_{\partial C} Dqf\right| \leq 8V\left(\left|h_1\right| + \ldots + \left|h_4\right|\right) \rho_n .$$

Since $\rho_n \to 0$ as $n \to \infty$, it follows that $\int_{\partial C} Dqf = 0$ □

Theorem 5.3 (Cauchy-Fueter integral formula for parallelepiped)**.** *If f is regular at every point of the positively oriented 4-parallelepiped C, and q_0 is a point in the interior of C,*

$$f\left(q_0\right)=\frac{1}{2\pi^2}\int_{\partial C}\frac{\left(q-q_0\right)^{-1}}{\left|q-q_0\right|^2}Dqf(q)$$

Proof. Theorem 4.1 provides us with the following:

$$Dq\wedge df_q=-\bar{\partial}_\ell f(q)v$$

where $v=dw\wedge dx\wedge dy\wedge dz$, for any differentiable function f. Similarly, we can also show that

$$df_q\wedge Dq=\partial_r f(q)v$$

Therefore, if f and g are both differentiable,then the following is also true

$$\begin{aligned}d(gDqf)&=d(gDq)f+gd(Dqf)\\&=dg\wedge Dqf-gDq\wedge df\\&=\left\{\left(\partial_r g\right)f+g\left(\partial_\ell f\right)\right\}v\end{aligned}$$

Now, suppose we allow $g(q)=\frac{(q-q_0)^{-1}}{|qq_0|^2}=\frac{q-q_0}{|q\quad q_0|^4}=\partial_r\left(\frac{1}{qq_0{}^2}\right)$; then g is differentiable at all points except at q_0. Its derivative is the following:

$$\bar{\partial}_r g=\Delta\left(\frac{1}{\left|q-q_0\right|^2}\right)=0.$$

If f is a regular function, then we have $\partial_\ell f=0$, hence

$$d\left[\frac{\left(q-q_0\right)^{-1}}{\left|q-q_0\right|^2}Dqf\right]=0$$

We can now follow the argument of theorem 4.2 to show the following:

$$\int_{\partial C'}\frac{\left(q-q_0\right)^{-1}}{\left|q-q_0\right|^2}Dqf(q)=0$$

where C' is any 4 -parallelepiped that does not contain q_0. If we dissect the given 4-parallelepiped C into 81 4-parallelepipeds with edges parallel to those of C, we can conclude that

$$\int_{\partial C}\frac{\left(q-q_0\right)^{-1}}{\left|q-q_0\right|^2}Dqf(q)=\int_{\partial C_0}\frac{\left(q-q_0\right)^{-1}}{\left|q-q_0\right|^2}Dqf(q),$$

where C_0 is any 4-parallelepiped containing q_0 that lies in the interior of C and has edges parallel to those of C. Now, take C_0 to have edge-vectors $\delta h_1,\ldots,\delta h_4$, where δ is a positive real number and $h_1,\ldots,h_4$ are the edgevectors of C, and allow q_0 to be at the centre of C_0 (so that the original vertex of C_0 is $q_0 - \frac{1}{2}\delta h_1 - \ldots - \frac{1}{2}\delta h_4)$; then

$$\min_{q\in\partial C_0} |q-q_0| = \min_{1\leq a,b,c\leq 4} \left| \frac{v\left(\delta h_1,\ldots,\delta h_4\right)}{Dq\left(\frac{1}{2}\delta h_a, \frac{1}{2}\delta h_b, \frac{1}{2}\delta h_c\right)} \right| = W\delta$$

where W depends only on $h_1,\ldots,h_4$. Because f is continuous at q_0, we can choose δ so that $q \in C_0\left(I^4\right) \Rightarrow |f(q) - f\left(q_0\right)| < \epsilon$ for any given $\epsilon > 0$; hence

$$\left| \int_{\partial C_0} \frac{\left(q-q_0\right)^{-1}}{\left|q-q_0\right|^2} Dq\left\{f(q) - f\left(q_0\right)\right\} \right| \leq \frac{8V}{W^3}\epsilon \tag{8}$$

where, as in theorem 4.2 .

$$V = \max_{1\leq a,b,c\leq 4} \left|Dq\left(h_a, h_b, h_c\right)\right|$$

Since the 3 -form $\frac{\left(q-q_0\right)^1 Dq}{\left|q-q_0\right|^2}$ is continuously differentiable and closed in $\mathbb{H}\backslash\left\{q_0\right\}$, based on Stokes's Theorem, we have

$$\int_{\partial C_0} \frac{\left(q-q_0\right)^{-1} Dq}{\left|q-q_0\right|^2} = \int_S \frac{\left(q-q_0\right)^{-1} Dq}{\left|q-q_0\right|^2}$$

where S is the 3 -sphere $|q-q_0| = 1$, oriented so that Dq is in the direction of the outward normal to S. Working in spherical coordinates (r,θ,ϕ,ψ), in which

$$q - q_0 = r\left(\cos\theta + i\sin\theta\cos\phi + j\sin\theta\sin\phi e^{-i\psi}\right)$$

we find that on S, i.e. when $r = 1$,

$$\begin{aligned} Dq &= \left(q-q_0\right)\sin^2\theta\sin\phi d\theta\wedge d\phi\wedge d\psi \\ &= \left(q-q_0\right) dS \end{aligned}$$

where dS is the usual Euclidean volume element on a 3 -sphere. Hence

$$\int_{\partial C_0} \frac{\left(q-q_0\right)^{-1} Dq}{\left|q-q_0\right|^2} = \int_S dS = 2\pi^2$$

Equation (9) then becomes the following:

$$\left| \int_{\partial C_0} \frac{\left(q-q_0\right)^1}{\left|q-q_0\right|^2} Dq f(q) - 2\pi^2 f\left(q_0\right) \right| \leq \frac{8V}{W^3}\epsilon$$

However, because we selected ϵ to be arbitrary, it follows that

$$\int_{\partial C} \frac{\left(q-q_0\right)^{-1}}{\left|q-q_0\right|^2} Dq f(q) = \int_{\partial C} \frac{\left(q-q_0\right)^{-1}}{\left|q-q_0\right|^2} Dq f(q) = 2\pi^2 f\left(q_0\right).$$

□

6 Special Relativity

One of the most intriguing applications of quaternions is a formulation of special relativity. The inventor of quaternions, Hamilton, had hoped that quaternions would be used to formulate physics. Over 60 years later, Einstein came up with special relativity, but used Minkowski's spacetime, which involved 4-vectors rather than quaternions. However, the quaternion formulation has a few advantages over this. Real quaternions form a division algebra and have positive quadratic norms, allowing us to use the same framework for rotations and translations of particles. Furthermore, multiplying two quaternions gives us a quaternion, as we saw in the beginning of this paper, but the same cannot be said for 4-vectors. The main reason why quaternions are a great candidate for formulating special relativity is that they can be described as a scalar plus a 3D vector, just like 4-vectors. This is a natural way to represent spacetime, as the vector corresponds to space, and the scalar corresponds to time.

Let us now describe the equivalence of quaternion time and space-time. We can define a quaternion time domain as a scalar clock at the origin of a 3D coordinate system, writing $\boldsymbol{t} = t_0 + \frac{\vec{x}}{c}$ where t_0 is the scalar clock, $\vec{x}$ is a space coordinate, and c is the speed of light. Dividing $\vec{x}$ by c gives us the time of light propagation. Now, if we place an observer at the origin, which is also where the clock is located. This means that the observer can precisely measure the scalar time of the clock at any given instant. Furthermore, if we had two simultaneous but different scalar clock signals, this observer could measure the time interval between them. Now suppose that the observer is at a quaternion location $\boldsymbol{t}$ and assume they are wearing a watch that has been perfectly synchronized with the clock. Orient the vector $\vec{x}$ such that it points from the clock towards the observer. Since the light from the clock will take some time to reach the observer, they will see different times on the watch and the clock. Since we have two simultaneous clock signals that are both scalar time images, the observer can calculate the quaternion time interval as such

$$\boldsymbol{\delta t} = \boldsymbol{t} - \boldsymbol{t_0} = t_0 + \frac{\vec{x}}{c} - t_0 = \frac{\vec{x}}{c}$$

, which is simply the time vector for light propagation. However, we want a time interval that doesn't involve direction. We want time to be a scalar quantity, that only depends on distance between the clock and observer. Multiplying by the conjugate of a quaternion can do just that. Thus, we have

$$\delta t = t - t_0 = \boldsymbol{\sqrt{\delta t \delta t^*}} = \sqrt{\frac{x^2}{c^2}} = \frac{x}{c}.$$

Note that if we switch the positions of the clock and the observer, we will still get $\frac{x}{c}$ as squared scalar quantities are always positive. Also note that the time interval can be used to calculate distance; $x = c\delta t$. Furthermore, if we wanted to calculate the time interval between two different signals from the origin, it would just be the difference between the scalar times, as there would be no light propagation delay.

What if the observer is moving (rotation or translation)? Then in the clock time interval $\delta t_0 = t_{02} - t_{01}$, the observer moves by $\delta\vec{x} = x_2 - x_1$. We can express the initial and final time values with quaternions:

$$\boldsymbol{t_1} = t_{01} + \frac{\vec{x_1}}{c}$$

$$\boldsymbol{t_2} = t_{02} + \frac{\vec{x_2}}{c}$$

And from this, we can calculate the quaternion time interval:

$$\boldsymbol{\delta t = t_2 - t_1} = t_{02} - t_{01} + \frac{\vec{x_2}}{c} - \frac{\vec{x_1}}{c} = \delta t_0 + \frac{\vec{\delta x}}{c}.$$

If we define the average velocity of the observing during our quaternion time interval as a quaternion $\boldsymbol{v} = \frac{\delta\vec{x}}{\boldsymbol{\delta t}}$, then we can express our time interval through the recursive formula:

$$\boldsymbol{\delta t} = \delta t_0 + \frac{\vec{\delta x}}{c} = \delta t_0 + \frac{\boldsymbol{v\delta t}}{c}.$$

Similarly, if the clock is moving, our interval is

$$\boldsymbol{\delta t} = \delta t_0 - \frac{\vec{\delta x}}{c} = \delta t_0 - \frac{\boldsymbol{v\delta t}}{c}.$$

Like when the observer was stationary, we want our answer to only depend on the magnitude of the vector, so we again multiply by the conjugate, giving

$$\delta t^2 = \boldsymbol{\delta t\overline{\delta t}} = \delta t_0^2 + \frac{\delta x^2}{c^2} = \delta t_0^2 + \frac{v^2\delta t^2}{c^2}.$$

Taking the square root gives us the Lorentz transformation for the moving time interval between both reference frames.

$$\delta t = \frac{\delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

This result is also present in traditional special relativity, but we have just derived it using the properties of quaternions. We've shown that the time intervals and measured time depend only on location and relative movement between the clock and observer, and that they are the same in both reference frames (clock and observer). We see that real quaternions represent physical quantities, and their absolute values correspond to the Lorentz transformations. See [1] [2] for an application of this formulation to mass, energy, and their Lorentz relations.

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