THE j-FUNCTION

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Abstract. This paper will explain the j-function and a few properties related to it. First, the definition and background is delegated as needed. Next, the concept of a modular function is introduced, as well as the corresponding q-expansion definition. Finally, the remainder of the paper is used to prove that j is modular and how its q-expansion is a key fact necessary to prove that $e^{\pi\sqrt{163}}$ is almost an integer.

1. Background

In class, when studying conformal mappings, we were able to touch briefly on linear fractional transformations - those of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

However, there are many things we can do beyond just conformal mappings that involve these transformations, among those being the j-function. To understand the basic idea behind what this "function" is, we need to recall a few background things first.

Definition 1.1. The group $SL_2(\mathbb{Z})$ is defined as the multiplicative group of all 2-by-2 integer matrices with determinant 1.

When we apply an element from $SL_2(\mathbb{Z})$ to an element in $\mathbb{C}$, we get a linear fractional transformation as described above. In order to understand what the j-function represents, we must now recall the concept of a lattice.

Definition 1.2. A *lattice* is an additive group generated by two elements in $\mathbb{C}$ that are not real multiples of one another.

In other words, a lattice L generated by $w_1, w_2 \in \mathbb{C}$ with $\frac{w_1}{w_2} \notin \mathbb{R}$ is written as

$$L = [w_1, w_2] = \{aw_1 + bw_2 : a, b \in \mathbb{Z}\}.$$

Example. The lattice $L = [1, i]$ generated by the imaginary and real units corresponds to the Gaussian integers (i.e. complex numbers of the form $a + bi$ where $a, b \in \mathbb{Z}$).

For an arbitrary lattice L, we will be interested in the following two functions, both of which are key parts in the final j-function:

$$g_2(L) = 60 \sum_{w \in L \setminus \{0\}} \frac{1}{w^4},$$

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$$g_3(L) = 140 \sum_{w \in L \setminus \{0\}} \frac{1}{w^6}.$$

Note that they are simply the 2^{nd} and 3^{rd} Eisenstein series with respect to L, respectively, just different because they have been multiplied by constants.

2. The j-function

Now we are almost ready to define the j-function, but let us start with the j-invariant.

Definition 2.1. The *j-invariant* of a lattice L is defined as

$$j(L) = 12^3 \cdot \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2}.$$

The denominator in the j-invariant is known as the discriminant of the lattice, denoted by $\Delta(L)$. One can show that this discriminant is never 0, which allows for the j-invariant to be well-defined for any lattice L. The only difference between the j-invariant and the j-function is that the former accepts a lattice as input while the latter accepts a single complex number. To fix this discrepancy, we define the following:

Definition 2.2. The *j-function* $j(\tau)$ is defined as

$$j(\tau) = j([1, \tau]),$$

where the right side represents the j-invariant of the lattice generated by 1 and τ.

Theorem 2.3. *The j-function is holomorphic on $\mathbb{H}$.*

This is a very important result of the j-function that can be proved by using the fact that both Eisenstein series of interest are holomorphic on $\mathbb{H}$ as well.

3. Modular Functions

Recall from chapter 14 of the notes that there are 6 possible congruences of matrices in $SL_2(\mathbb{Z})$, and that only 2 are needed to generate the entire group. We let these two matrices be

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that applying T to any $z \in \mathbb{H}$ will result in a translation of $+1$ while applying S will result in the negative reciprocal, namely $-\frac{1}{z}$.

Using the matrices S and T we can limit our analysis on $\mathbb{H}$ to a fundamental region F. We define this region F such that any point outside of F can be translated using our two matrices such that the image lies in F. Each point in F must also be unique, so that they

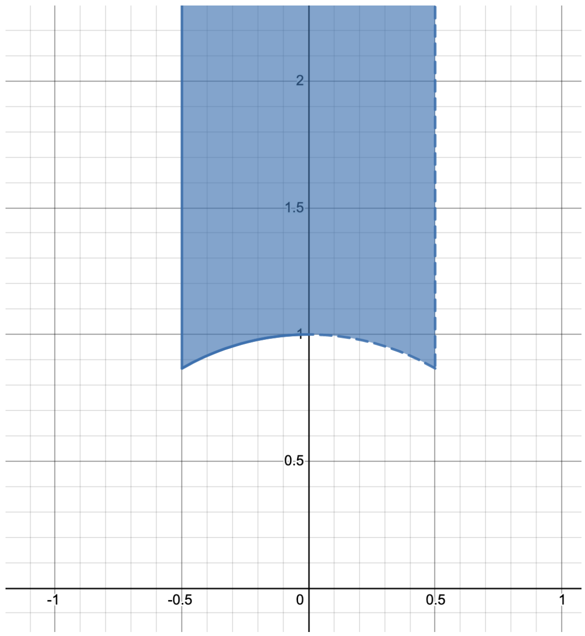

Figure 1. A visualization of our choice of F.

cannot be mapped onto each other after a series of transformations by S and T.

Finding such a region is not hard: the matrix T limits us to finding a single strip of width 1 parallel to the imaginary axis, and matrix S limits us to including points that lie outside of the unit disc. Our final choice of F can be the unit strip $-\frac{1}{2} \leq \Re(z) < \frac{1}{2}$ excluding the unit disk ($|z| > 1$). As an edge case, we also must include the unit circle from $z = -\frac{1}{2}$ to $z = 0$ inclusive. A visualization of this region is shown in Figure 1.

Now that we have narrowed down the region of analysis for the functions of interest, we are ready to define a new term.

Definition 3.1. Let $f : \mathbb{H} \to \mathbb{C}$ be a function which satisfies the following properties:

- f is meromorphic on $\mathbb{H}$.
- f is invariant under any transformation in $SL_2(\mathbb{Z})$. In other words, $f(\gamma \cdot \tau) = f(\tau)$ for any $\gamma \in SL_2(\mathbb{Z})$.
- $f(z)$ has a Fourier expansion as $\Im(z)$ goes to infinity, namely in the form of a Laurent series

$$f(\tau) = \sum_{n=-m}^{\infty} a(n) q^n$$

where $q = e^{2\pi i \tau}$.

Then we call f a *modular function.*

Remark 3.2. The second property is the one that allows us to only look at our region F, hence the name "modular". A more general term called a *modular form* has a modified version of this property, where $f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

To show that j is indeed also a modular function, we must prove the second and third properties of Definition 3.1. We begin with the following theorem:

Theorem 3.3. *If L and L' are lattices in $\mathbb{C}$, then $j(L) = j(L')$ if and only if $L = \lambda L'$ for some $\lambda \in \mathbb{C}$.*

Proof. For the forward direction, the basic idea for the reverse direction is the compare the Weierstrass $\wp$ functions between the two lattices. Since they are the same, the two have the same poles. Next, by some casework and manipulation from the original j-function identity $j(L') = j(L)$ we can see that the lattices are indeed homothetic. A full proof is given in [Sch10].

Now we show the simpler reverse direction. Assume that $L = \lambda L'$ for some $\lambda \in \mathbb{C}$. Then we can compute

$$g_2(L) = g_2(\lambda L') = 60 \sum_{\omega \in L' \setminus \{0\}} \frac{1}{(\lambda\omega)^4} = \frac{1}{\lambda^4} 60 \sum_{\omega \in L' \setminus \{0\}} \frac{1}{\omega^4} = \frac{g_2(L')}{\lambda^4}.$$

Similarly,

$$g_3(L) = \frac{g_3(L')}{\lambda^6}.$$

Now we can use direct substitution to show what we want. Starting with $j(L)$, we have

$$\begin{aligned} j(L) &= 1728 \cdot \frac{g_2(L)^3}{g_2(L)^3 - 27 g_3(L)^2} \\ &= 1728 \cdot \frac{\frac{g_2(L')^3}{\lambda^{12}}}{\frac{g_2(L')^3}{\lambda^{12}} - 27 \cdot \frac{g_3(L')^2}{\lambda^{12}}} \\ &= 1728 \cdot \frac{g_2(L')^3}{g_2(L')^3 - 27 g_3(L')^2} = j(L'), \end{aligned}$$

as desired.

□

We now turn to completing the proof of the second property, namely being that $j(\tau)$ is $SL_2(\mathbb{Z})$-invariant.

Theorem 3.4. *If $\tau, \tau' \in \mathbb{H}$, then $j(\tau) = j(\tau')$ if and only if $\tau' = \gamma\tau$ for some $\gamma \in SL_2(\mathbb{Z})$.*

Proof. We first prove the forwards direction, assuming that $j(\tau) = j(\tau')$. Using Theorem 3.3, we see that the lattices $[1, \tau]$ and $[1, \tau']$ must be homothetic to each other, or in other words, one is a complex multiple of the other. However, that means we should be able to express the lattice $[\lambda, \lambda\tau]$ for some $\lambda \in \mathbb{C}$ solely using 1 and τ', which we do by letting

$$\lambda = r\tau' + s \text{ and } \lambda\tau = p\tau' + q,$$

and the 4 coefficients are all integers. Now we use some matrices to express this. We have

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda\tau \\ \lambda \end{pmatrix},$$

and conversely, we can also find some $a, b, c, d \in \mathbb{Z}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda\tau \\ \tau \end{pmatrix} = \begin{pmatrix} \tau' \\ 1 \end{pmatrix}.$$

Combining these two equations gives us

$$A \begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \begin{pmatrix} \tau' \\ 1 \end{pmatrix}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, so if we let $A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, we have $a'\tau' + b' = \tau'$, or that

$$a' + b'\frac{1}{\tau'} = 1.$$

However, since τ' and 1 are not real multiples of each other, we must have $a' = 1$ and $b' = 0$. This implies that A is the identity matrix and hence

$$\det(A) = 1 = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

We quickly check that

$$\Im(\tau) = \Im\left(\frac{p\tau' + q}{r\tau' + s}\right) = \frac{b(ps - qr)}{|r\tau + s|^2} = \frac{\Im(\tau')(ps - qr)}{|r\tau + s|^2},$$

which implies that $ps - qr > 0$ since both imaginary parts are positive for τ and τ' (this follows from both being in $\mathbb{H}$).

Recalling the determinant equation from above, since both determinants on the right are integral, $\det \begin{pmatrix} p & q \\ r & s \end{pmatrix} = 1$ and hence $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z})$, as claimed.

Now we turn to the other direction of the proof, beginning with the supposition that $\tau' = \frac{p\tau+q}{r\tau+s}$ for $p, q, r, s \in \mathbb{Z}$ with $ps - qr = 1$.

Let $\lambda = r\tau + s$. Then

$$\lambda[1, \tau'] = (r\tau + s)\left[1, \frac{p\tau + q}{r\tau + s}\right] = [r\tau + s, p\tau + q].$$

However, we can express $\tau = -q(r\tau + s) + s(p\tau + q)$ and $1 = p(r\tau + s) - r(p\tau + q)$, which means that each element in $[1, \tau]$ may be expressed in terms of λ and $\lambda\tau'$. However, we also know that λ and $\lambda\tau'$ can also be expressed using just 1 and τ, which means that both lattices $\lambda[1, \tau']$ and $[1, \tau]$ contain each other. As a result, we must have

$$[r\tau + s, p\tau + q] = \lambda[1, \tau'] = [1, \tau].$$

Using Theorem 3.3, we can see that this implies $j(\tau) = j(\tau')$ so we are done. □

We now turn to proving the third property regarding the existence of a q-expansion for the j-function. We shall do this by showing that $\Delta([1, \tau]) = g_2(\tau)^3 - 27g_3(\tau)^2$ goes to 0 as $\Im(\tau) \to \infty$.

We start by considering any number in $g_2(\tau)$ that includes a nonzero multiple of τ in the denominator, say $\frac{1}{(m+n\tau)^4}$. As $\Im(\tau) \to \infty$, we see that

$$\frac{1}{(m+n\tau)^4)} = \frac{1}{m^2 + 4m^3(n\tau) + 6m^2(n\tau)^2 + 4m(n\tau)^3 + (n\tau)^4}$$

will be dominated by the final $(n\tau)^4$ term which will cause it to go to 0. Then, when computing $g_2(\tau)$, we only need to consider lattice points that do not include multiples of τ, which are namely the nonzero integers. Hence we have

$$\lim_{\Im(\tau)\to\infty} g_2(\tau) = 60 \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{m^4} = 120 \sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{4\pi^4}{3}.$$

Similarly, we only need to consider integer terms for $g_3(\tau)$:

$$\lim_{\Im(\tau)\to\infty} g_3(\tau) = 140 \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{m^6} = 280 \sum_{m=1}^{\infty} \frac{1}{m^6} = \frac{8\pi^6}{27}.$$

As a result, we have

$$\lim_{\Im(\tau)\to\infty} (g_2(\tau)^3 - 27g_3(\tau)^2) = \left(\frac{4\pi^4}{3}\right)^3 - 27\left(\frac{8\pi^6}{27}\right)^2 = 0.$$

This implies that $\Delta([1,\tau])$ has a isolated zero at ∞ and hence $j(\tau)$ has a pole there as well. This completes the proof that $j(\tau)$ is meromorphic and hence has a q-expansion.

To find the coefficients of this particular expansion, one utilizes the Fourier series of $g_2(L)$ and $g_3(L)$. From here, the calculations show that the q-expansion begins as follows:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots .$$

Surprisingly enough, all coefficients in this expansion are integers. Combining this existence of the q-expansion with $SL_2(\mathbb{Z})$-invariance results in the desired corollary.

Corollary 3.5. *The j-function is a modular function.*

4. Ramanujan's Constant

Now we will explore a very interesting result regarding the strange constant $e^{\pi\sqrt{163}}$. To do so, we need the fact that the imaginary quadratic field $\mathbb{Q}(\sqrt{-163})$ has class number 1. This basically means that any number in the ring of integers $\mathbb{Z}\left[\frac{1+\sqrt{-163}}{2}\right]$ has exactly one unique factorization in the ring.

Next, we can deduce from some basic properties of elliptic functions the following. Before doing so, however, let us examine the basic idea behind what we call complex multiplication.

Definition 4.1. Let E be an elliptic curve in $\mathbb{C}$. We say that E has *complex multiplication* if it has an endomorphism ring $\mathrm{End}(E)$ that is greater than the integers $\mathbb{Z}$.

The basic idea of complex multiplication that is useful here can be summarized as such: if K is an imaginary quadratic number field that has a ring of integers denoted by A, and L is a lattice in $\mathbb{C}$ so that L is also an ideal of A, the elliptic curve $E = \mathbb{C}/L$ has complex multiplication. This can be seen because L is an ideal and hence it is preserved by multiplication. As a result, $\text{End}(E) = A$, which is larger than $\mathbb{Z}$.

Due to this correspondence, the conjugates of the j-function $j(E)$ have the form $j(E')$, where E' is another elliptic curve that has complex multiplication by the ring A.

This lays the foundation for our main proposition.

Proposition 4.2. $j\left(\frac{-1+\sqrt{-163}}{2}\right)$ *is an algebraic integer of degree at most* 1.

This proposition follows from a much stronger theorem in complex multiplication, which is documented in [Sur] and [Sil94] but is too complicated to be explained here. Here, note that an algebraic integer of degree n is a zero of a monic polynomial of degree n with integer coefficients. Thus, we can deduce that because the degree of $j(\tau)$ for any $\tau \in \mathbb{Q}(\sqrt{-163})$ is equal to the class number of the field, which is 1,

$$j\left(\frac{-1+\sqrt{-163}}{2}\right) + a_0 = 0$$

for some integer a_0, implying that $j\left(\frac{-1+\sqrt{-163}}{2}\right) \in \mathbb{Z}$.

Now referring to the q-expansion once again, if we plug in a very small q into the expression, many of the later terms will be very close to 0. We explore what this means for the value $j\left(\frac{-1+\sqrt{-163}}{2}\right)$ in particular, first by computing q in this case:

$$q = e^{2\pi i\tau} = e^{2\pi i\left(\frac{-1+\sqrt{-163}}{2}\right)} = e^{-\pi i-\pi\sqrt{163}} = -e^{-\pi\sqrt{163}}.$$

Looking at just a few early terms in the q-expansion, we see that

$$j\left(\frac{-1+\sqrt{-163}}{2}\right) = \frac{1}{-e^{-\pi\sqrt{163}}} + 744 + 196884\left(-e^{-\pi\sqrt{163}}\right) + 21493760\left(-e^{-\pi\sqrt{163}}\right)^2 + \cdots \in \mathbb{Z}.$$

Only the first two terms, one of which is an integer already, will have a significant impact on the final number, so we have (to a very accurate degree)

$$\frac{1}{-e^{-\pi\sqrt{163}}} + 744 = \text{integer} + \mathcal{O}(e^{-\pi\sqrt{163}}).$$

We do indeed find that $e^{\pi\sqrt{163}} \approx 262537412640768743.99999999999992$, which is referred to as *Ramanujan's constant.*

While this is a number very close to an integer, there are also a few more numbers which are close to integers for similar reasons. For example, the next two imaginary quadratic fields with class number 1 are $\mathbb{Q}(\sqrt{-67})$ and $\mathbb{Q}(\sqrt{-43})$. As one would expect, both $e^{\pi\sqrt{67}} = 147197952743.9999987$ and $e^{\pi\sqrt{43}} = 884736743.999777$ are both relatively close to whole numbers as well, albeit to a larger difference than Ramanujan's constant (we can credit this to the fact that the later terms in the q-expansion do not vanish as quickly).

References

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