FOURIER SERIES

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1. Fourier Series

Indeed, any periodic function, even the ones with discontinuities, can be represented as a series of sum of $\sin(nx)$ and $\cos(nx)$ for $n \in \mathbb{N}$, which, together, is called a Fourier series. A Fourier series has the complex form $\sum_{n=-\infty}^{\infty} a_n e^{inx}$. To find the coefficients a_n, we use the following formula, which is derived using the orthogonality of trignometric functions

$$a_k = \frac{1}{2 \cdot \pi} \int_{-\pi}^{\pi} \frac{f(x)}{e^{ikx}} dx$$

In general, for functions with period $2L$, the Fourier series coefficients can be calculated through the integral,

$$a_k = \frac{1}{2L} \int_{-L}^{L} \frac{f(x)}{e^{\frac{kix\pi}{L}}} dx$$

.

While Fourier series only applies to periodic function, we can try a little bit harder to generalize it.

Example. Show that $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

Given the function $f(x) = x^2$ on the interval $-1 \leq x \leq 1$. We can use the integral

$$\frac{1}{2} \int_{-1}^{1} \frac{x}{e^{kix\pi}} dx = \frac{1}{2} \int_{-1}^{1} x \cdot (\cos(kx\pi) - i \cdot \sin(kx\pi)) dx$$

to find the coefficients a_k. Using integration, we can see that the coefficient is equal to $a_k = (-1)^{k+1} \frac{i}{k\pi}$. Now, we can write

$$x = \sum_{1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

for $-1 \leq x \leq 1$. Plug in $x = \frac{1}{2}$, we have

$$\frac{1}{2} = \sum_{1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(\frac{n\pi}{2}) = \sum_{0}^{\infty} \frac{2(-1)^n}{\pi(2n+1)}$$

Moving the $\frac{2}{\pi}$ to the left, we are left with the summation $\frac{\pi}{4} = \sum_{0}^{\infty} \frac{(-1)^n}{2n+1}$.

2. Fourier Transform

The idea of Fourier Transform is to assume that we have a periodic function whose period is ∞. Thus, we can write down the Fourier series of the function $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{inx\pi}{L}} = \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} \frac{f(x)}{e^{\frac{nix\pi}{L}}} dx \; e^{\frac{inx\pi}{L}}$ according to the formula from Fourier series. We use a change of variable $k = \frac{n \cdot \pi}{L}$ and $\Delta k = \frac{\pi}{L}$, then $f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} \int_{-L}^{L} \frac{f(x)}{e^{kix}} dx \; e^{ikx}$.
Taking the limit as $L \to \infty$, we get $f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(x)}{e^{ikx}} dx \; e^{ikx} \; dk$. The effect of the two integration cancels out and turn out to be the forward and inverse Fourier Transform respectively.

Thus, the Fourier Transform of a function is

$$F(k) = \int_{-\infty}^{\infty} \frac{f(x)}{e^{ikx}} dx$$

The inverse Fourier Transform is

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} F(k) e^{ikx} dk$$

Since $k = \frac{n \cdot \pi}{L}$, when L represents the period of a function, k, which is the reciprocal of L, represents the frequency from a physics point of view.

Properties of Fourier Transform:

A shift in time domain $f(t - t_0)$ under Fourier Transform would lead to an exponential term: $\mathscr{F}(f(t - t_0)) = F(x) * e^{-ik(t_0)}$.

A shift in frequency domain $F(x - x_0)$ under the Inverse Fourier Transform would also lead to an exponential term: $\mathscr{F}^{-1}(F(k - k_0)) = f(t)e^{itk_0}$

Convolution:

Convolution is defined as follow: Let $f(t)$ and $g(t)$ be two functions in the time domain, then the convolution of the two function, $f(t) * g(t)$ is equivalent to

$$\int_{-\infty}^{\infty} g(t') f(t - t') dt'$$

If we Fourier-transform the integral, the result is $\mathscr{F}(f(t) * g(t)) = \mathscr{F}(f(t)) \cdot \mathscr{F}(g(t))$. We will prove this by showing that the $\mathscr{F}^{-1}(\hat{f}(x) \cdot \hat{g}(x)) = f(t) * g(t)$.

Proof:

$$\mathscr{F}^{-1}(\hat{f}(x) \cdot \hat{g}(x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x) \int_{-\infty}^{\infty} g(y) e^{-iyx} dy e^{ixt} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(x)g(y)e^{i(t-y)x}dydx$$
$$= \int_{-\infty}^{\infty}g(y)\int_{-\infty}^{\infty}\frac{1}{2\pi}\hat{f}(x)e^{i(t-y)x}dxdy$$
$$= \int_{-\infty}^{\infty}g(y)f(t-y)dy$$
$$= \int_{-\infty}^{\infty}f(y)g(t-y)dy$$

3. Parseval's Identity

We first state the Parseval's Identity. If $f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{in\pi x}{L}}$, then

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{-\infty}^{\infty}|c_n|^2$$

Proof: Let

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{in\pi x}{L}} = \sum_{m=-\infty}^{\infty} c_m \cdot e^{\frac{im\pi x}{L}}$$

It's easy to see that

$$|f(x)|^2 = f(x)\cdot\overline{f(x)} = \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{in\pi x}{L}} \cdot \sum_{m=-\infty}^{\infty} \overline{c_m}\cdot e^{\frac{-im\pi x}{L}} = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n\cdot\overline{c_m}e^{\frac{i(n-m)\pi x}{L}}$$

. Taking the integral, we have

$$\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n\cdot\overline{c_m}\int_{-L}^{L}e^{\frac{i(n-m)\pi x}{L}}dx$$

We know that unless $n = m$, $\int_{-L}^{L} e^{\frac{i(n-m)\pi x}{L}}dx$ will be zero. Thus, we can simplify the summation by eliminating terms whose $n \neq m$.

$$\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty} 2L|c_n|^2$$

4. Poisson Summation Formula

The Fourier Transform of a function and function itself exhibit surprising properties, one of which is demonstrated through the Poisson Summation Formula:

$$\sum_{-\infty}^{\infty} f(x+n) = \sum_{-\infty}^{\infty}\hat{f}(x+n)$$

Proof: Let $f(x)$ be defined on $\mathbb{R}$ and $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$, then since $F(x)$ has a period of 1, we can find its Fourier series: $F(x) = \sum_{n=-\infty}^{\infty}\int_{n}^{n+1} F(x)\ dx\ e^{-ikx} = \int_{-\infty}^{\infty} F(x)\ dx$

$e^{-ikx} = \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x+n)\ dx\ e^{-ikx} = \sum_{-\infty}^{\infty} \hat{f}(x+n)$. But we note that $F(x)$ is merely $\sum_{-\infty}^{\infty} f(x+n)$. Thus, we've shown that

$$\sum_{-\infty}^{\infty} f(x+n) = \sum_{-\infty}^{\infty} \hat{f}(x+n)$$

This also requires $\sum_{n=-\infty}^{\infty} f(x+n)$ to be convergent.

5. Convergence of Fourier Transform and the Laplace Transform

Looking at the Fourier Transform, $F(k) = \int_{-\infty}^{\infty} \frac{f(x)}{e^{ikx}} dx$, a question would naturally arise: What happen when the integral diverges? Or, does all function have a Fourier Transform? For example, functions such as $e^{\lambda x}$ and $\sin(x)$, give us headache when we try to find its Fourier Transform. Luckily, an alternative to this dilemma would be a "different" sort of Fourier Transform, called the Laplace Transform.

Let's try to multiply the badly-behaved function $f(x)$ by an exponential $e^{-\lambda t}$ such that the integrand $f(x) \cdot e^{-(\lambda+ik)x}$ converges. However, when $x \to -\infty$, the exponential gets arbitrarily large, so we tackle this problem by multiplying the integrand by a function $H(x)$, where $H(x) = 1$ when $x > 0$, and $H(x) = 0$ when $x \leq 0$. Thus, the Fourier Transform now becomes $F(k) = \int_0^{\infty} f(x) \cdot e^{-(\lambda+ik)x} dx$. We rename the variable $s = \lambda + ik$, so it becomes $F(s) = \int_0^{\infty} f(x) \cdot e^{-sx} dx$. Using the formula for inverse Fourier Transform, we get that the inverse Laplace Transform is $\frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\lambda - i\omega}^{\lambda + i\omega} F(s) e^{st} ds$.

6. Reference

https://lpsa.swarthmore.edu/Fourier/Xforms/FXformIntro.html

https://www.thefouriertransform.com/

https://math.mit.edu/ gs/cse/websections/cse41.pdf