The Wiener-Ikehara Theorem

Justin French

May 29, 2016

1 Overview

The Wiener-Ikehara Theorem allows one to find asymptotic properties of certain number theoretic functions based on properties of the Riemann zeta function and related Dirichlet series. For example, using just the Wiener-Ikehara Theorem and the fact that the Riemann zeta function has no zeroes on the line $\Re(s) = 1$, one can deduce the prime number theorem

$$\pi(x) \sim \frac{x}{\log x}$$

where $\pi(x)$ is the prime counting function and the symbol $\sim$ means that the ratio of each side approaches one as x approaches infinity. Here, we will state the theorem, prove a special case of it, and then show some applications.

2 Statement of the Theorem

Stated for Dirichlet Series, the Wiener-Ikehara Theorem is as follows:

Theorem: Let $F(s)$ be a Dirichlet series given by

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

If $f(n) \geq 0$ for all n, $F(s)$ converges on the open half plane $\{z \in \mathbb{C} : \Re(z) > 1\}$, and $F(s)$ can be continued to a function holomorphic on an open set containing the corresponding closed half plane except for possibly a simple pole as $s = 1$, then

$$\sum_{n \leq x} f(n) \sim Ax$$

where A is the residue of $F(s)$ at $s = 1$.

Note that, in the requirements listed for the Wiener-Ikehara Theorem to be applicable, there is no growth condition on $F(s)$. Most similar theorems require that $F(s)$ obey some asymptotic law either on the line $\Re(s) = 1$ or elsewhere, so the fact that this theorem does not is exceptional and makes it easier to use in practice. This theorem is often stated in a more general form involving the Laplace Transform:

Theorem: Let $A(x)$ be non-negative and non-decreasing on $[0, \infty)$. If

$$F(s) = \int_0^\infty A(x) e^{-xs} dx$$

converges for $\Re(s) > 1$ and can be continued for $\Re(s) \geq 1$, except for possibly a simple pole at $s = 1$, then

$$A(x) \sim Ae^{-x}$$

where again A is the residue of $F(s)$ at $s = 1$.

The first statement can be seen to be a special case of this one by putting $A(e^x) = \sum_{n \leq x} f(n)$. Though this form is more general, the Dirichlet series form is closer to what would be used in practice. There is a general form for Dirichlet series which is applicable for poles of higher order.

Theorem: Let $F(s)$ be a Dirichlet series given by

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

Suppose that the series converges for $\Re(s) > \alpha$, and that it can be analytically continued to $\Re(s) \geq \alpha$ except for possibly a pole of order γ at α with $(x - \alpha)^{-\gamma}$ coefficient A. Then

$$\sum_{n \leq x} f(n) \sim \frac{Ax^\alpha \log(x)^{\gamma - 1}}{\alpha(\gamma - 1)!}$$

3 Proof (omitted)

The proof of the Wiener-Ikehara Theorem is not very interesting; it is mostly just evaluating a lot of integrals and Fourier transforms, and it does not provide great insight as to why the theorem works. As such, the proof of the theorem will be omitted, but different proofs of the theorem can be found partially or wholly in any of the resources listed at the end of the text.

4 Example 1: The Prime Number Theorem

One of the major results of the Wiener-Ikehara Theorem is a simple proof prime number theorem. As mentioned before, the prime number theorem states that

$$\pi(x) \sim \frac{x}{\log(x)}$$

Since $\pi(x) = \sum_{n \leq x} 1_p(n)$ where $1_p(n) = 1$ if n is prime and 0 otherwise, one might be tempted to use the prime zeta function $\sum_{n=1}^{\infty} \frac{1_p(n)}{n^s}$for proving the prime number theorem. However, this function cannot be continued at all to the line $\Re(s) = 1$, so the Wiener-Ikehara Theorem cannot be applied (this fact is essentially due because the primes are not very close together).

Instead, we will use the Von Mangoldt function $\Lambda(n)$ as the coefficients of the Dirichlet series. This function is given by

$$\Lambda(n) = \begin{cases} \log(p) & n = p^k \\ 0 & else \end{cases}$$

We also define the Chebyshev function

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

We will use the Wiener-Ikehara Theorem to find an asymptotic expression for this function, and then we will prove that that asymptotic expression implies the prime number theorem.

First, we find a nicer expression for the Dirichlet Series

$$F(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Consider the Euler product for the Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

Taking logarithms, we get

$$\log(\zeta(s)) = -\sum_p \log(1-p^{-s})$$

and differentiating:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\log(p)p^{-s}}{1-p^{-s}}$$

Moving the negative sign and expanding the power series, one obtains

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{n=1}^{\infty} \log(p)p^{-ns}$$

Notice that every term in the sum is a prime power with the coefficient $\log(p)$. This sum is exactly the definition of $F(s)$, only rearranged slightly, so

$$F(s) = -\frac{\zeta'(s)}{\zeta(s)}$$

Now, we apply the Wiener-Ikehara theorem. Since $\zeta(s)$ has no zeroes on the line $\Re(s) = 1$, the function $F(s)$ can be extended to the closed half-plane except for a simple pole at $s = 1$. Hence, all of the criteria of the Wiener-Ikehara Theorem is satisfied, and it only remains to calculate the residue at $s = 1$.

$$\zeta(s) = \frac{1}{s-1} + ...$$

so

$$\zeta'(s) = -\frac{1}{(s-1)^2} + ...$$

Simply dividing shows that the residue at $s = 1$ is 1, so $\psi(x) \sim x$.

To show the prime number theorem, we use some simple inequalities:

$$\frac{\psi(x)}{x} = \frac{1}{x}\sum_{p\leq x} \log(p) \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \leq \frac{1}{x}\sum_{p\leq x} \log(x) = \frac{\pi(x)}{x/\log(x)}$$

and for any $\epsilon > 0$

$$\frac{\psi(x)}{x} \geq \frac{1}{x} \sum_{x^{1-\epsilon} \leq p \leq x} \log(p) \geq \frac{1}{x} \sum_{x^{1-\epsilon} \leq p \leq x} \log(x^{1-\epsilon}) = \frac{1-\epsilon}{x} \sum_{x^{1-\epsilon} \leq p \leq x} \log(x)$$
$$= \frac{(1-\epsilon)(\pi(x) - \pi(x^{1-\epsilon}))}{x/\log(x)} \geq \frac{(1-\epsilon)\pi(x)}{x/\log(x)} - (1-\epsilon)x^{-\epsilon}\log(x)$$

where the last inequality is due to $\pi(x^{1-\epsilon}) \leq x^{1-\epsilon}$. Sending x to infinity and using $\psi(x) \sim x$, we get

$$\frac{1}{1-\epsilon} \geq \lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)}$$

Combining this inequality with the first, we see

$$1 \leq \lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} \leq \frac{1}{1-\epsilon}$$

Since ϵ can be arbitrarily small, we obtain the prime number theorem:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1$$

5 Example 2: Euler's Totient Function

Define Euler's totient function $\phi(n) = |\{k \leq n : \gcd(k, n) = 1\}|$. Colloquially, this function counts the numbers less than or equal to n that are relatively prime to n. ϕ is a multiplicative function, which means that $\phi(ab) = \phi(a)\phi(b)$ whenever a and b are relatively prime. Also, $\phi(p^k) = p^{k-1}(p-1)$ for primes p.

One question that one might ask about this function is this: on average, how many times does given some number a equal $\phi(n)$ for some n? This question can be answered with the Wiener-Ikehara Theorem. Define $a(n) = |\{k : \phi(k) = n\}|$ The Dirichlet series for $a(n)$ is

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{\phi(n)^s}$$

Since ϕ is multiplicative, it can be expressed with the following Euler product:

$$F(s) = \prod_p \sum_{k=0}^{\infty} \frac{1}{\phi(p^k)^s}$$

Separating $\phi(p^0) = 1$ and replacing $\phi(p^k)$ with $p^{k-1}(p-1)$, we get

$$F(s) = \prod_p (1 + \sum_{k=1}^{\infty} \frac{1}{p^{(k-1)s}(p-1)^s})$$

Factoring out $\frac{1}{(p-1)^s}$ and using the geometric series formula,

$$F(s) = \prod_p (1 + \frac{1}{(p-1)^s} \frac{1}{1-p^-s})$$

Finally, we multiply by $\frac{\zeta(s)}{\zeta(s)}$ to put the equation in a more manageable form:

$$F(s) = \zeta(s) \prod_p (1 - p^{-s} + (p-1)^{-s})$$

Because the sum $\sum_{n=1}^{\infty} (\frac{1}{n^s} - \frac{1}{(n+1)^s})$ converges for $\Re(s) \geq 1$, $F(s)$ is holomorphic on $\Re(s) \geq 1$ except for the simple pole at $s = 1$ from $\zeta(s)$. Hence, we can apply the Wiener-Ikehara Theorem.

The residue at $s = 1$ of $F(s)$ will be

$$\lim_{s \to 1} (s-1)\zeta(s) \prod_p (1 - p^{-s} + (p-1)^{-s}) = \prod_p (1 - p^{-1} + (p-1)^{-1})$$

Let $A = \prod_p (1 - p^{-1} + (p-1)^{-1})$. Then

$$\begin{aligned}
A \prod_p (1 - p^{-2} - p^{-3} + p^{-5}) &= \prod_p (1 - p^{-1} + (p-1)^{-1})(1 - p^{-2} - p^{-3} + p^{-5}) \\
&= \prod_p (1 - p^{-2} - p^{-3} + p^{-5} - p^{-1} + p^{-3} + p^{-4} - p^{-6} + \\
&\quad (p-1)^{-1} - (p-1)^{-1}p^{-2} - (p-1)^{-1}p^{-3} + (p-1)^{-1}p^{-5}) \\
&= \prod_p (1 - p^{-1} - p^{-2} + p^{-4} + p^{-5} - p^{-6} + p^{-2}\frac{p^2-1}{p-1} - p^{-5}\frac{p^2-1}{p-1} \\
&= \prod_p (1 - p^{-1} - p^{-2} + p^{-4} + p^{-5} - p^{-6} + p^{-1} + p^{-2} - p^{-4} - p^{-5}) \\
&= \prod_p (1 - p^{-6})
\end{aligned}$$

Rearranging and factoring, we get

$$A = \prod_p \frac{1 - p^{-6}}{(1-p^{-2})(1-p^{-3})} = \frac{\frac{1}{\zeta(6)}}{\frac{1}{\zeta(2)}\frac{1}{\zeta(3)}} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \approx 1.944$$

Therefore, by the Wiener-Ikehara Theorem,

$$\sum_{n \leq x} a(n) \sim \frac{\zeta(2)\zeta(3)}{\zeta(6)} x$$

and consequently

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} a(n) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}$$

so $\frac{\zeta(2)\zeta(3)}{\zeta(6)}$ is the average value of $a(n)$.

6 Resources

Divergent Series by G. H. Hardy

Introduction to Analytic Number Theory by K. Chandrasekharan

Analytic Number Theory: An Introductory Course by Paul T. Bateman and Harold G. Diamond

http://dare.uva.nl/document/2/41210

http://www.math.leidenuniv.nl/~evertse/ant13-6.pdf