BROWNIAN MOTION

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1. An Introduction to an Introduction to Brownian Motion

Brownian motion is very immense, the idea behind it allowing for proofs of every theorem as some claim. To visualize it, let us start with a metaphor, imagine a football stadium with a large crowd. A very large balloon lies on top of many members of the crowd at a single moment. These fans, due to excitement hit the balloon at different times and in different directions with the choice of direction being completely random, but all with the same strength. Consider now the force exerted at a certain time. The balloon will move in the direction that it is hit. Now look at the balloon from far above, so that we cannot see the supporters. We can see the large balloon as a small object animated by erratic movement. Another commonly used metaphor is that of a drunkard's walk, who can randomly move left or right at discrete, equi-spaced instances of time. This random movement also mimics Brownian motion.

Stochastic calculus (random calculus) helps in working with non-deterministic functions. It is necessary, but it is part of the nitty-gritties, so we will explain what we need as we get to it.

2. Introduction to Brownian Motion

Consider the symmetric random walk, in which in each unit time we are equally likely to take a step to the left or to the right. Now imagine that we speed up this process by taking smaller and smaller steps in smaller and smaller time intervals. The limit is what we know as Brownian motion.

Say that for each Δt time unit, we randomly take a step of Δx to the left or right. We can say that the position at a time, $X(t)$ is:

$$X(t) = \Delta x(X_1 + X_2 + ... + X_{(t/\Delta t)})$$

Where X_i is the "state" at time t_i: +1 if the step of Δx is to the right and -1 if to the left. The X_i are assumed independent and the two states have equal probabilities.

We now have that $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X(t)] = 0$, which means that there is no expected function-this motion is indeed random. Further, we can look at the variation:

$$Var(X(t)) = \Delta x^2(t/\Delta t)$$

We now let Δx and Δt go to 0. However, we cannot let them equal each other as that case is trivial ($\mathbb{E}[X(t)]$ and $Var(X(t))$ would both converge to 0 and thus $X(t)$ would be 0 with probability 1).

We can let $\Delta x = c\sqrt{\Delta t}$ for some positive constant c, then:

$$E[X(t)] = 0$$

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$$Var(X(t)) \to c^2 t$$

Since the variance increases with time, and is non-zero, we can see that Brownian Motion is random and increasing in randomness.

Definition 2.1 (Brownian Motion)**.** A stochastic process is said to be a Brownian Motion process if

(1) $X(0) = 0$
(2) $\{X(t), t \geq 0\}$ has stationary and independent increments
(3) For every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $c^2 t$

A standard Brownian motion is one where c = 1.

3. Probability Theory

Because Brownian motion is a continuous-time stochastic process, there is some indeterminancy as to how the motion proceeds. Thus, some probability theory must be covered to fully understand Brownian motion.

Definition 3.1 (Probability Space)**.** A probability space is a triple$(\Omega,\ \mathcal{F},\ P)$ consisting of:

- the sample space Ω, the set of all possible outcomes, as an arbitrary non-empty set
- the σ-algebra $\mathcal{F} \subseteq 2^{\Omega}$, where 2^{Ω} is the power set of Ω, such that:
  - $\mathcal{F}$ contains the sample space: $\Omega \in \mathcal{F}$
  - $\mathcal{F}$ is closed under complements: if $A \in \mathcal{F}$, then $(\Omega \backslash A) \in \mathcal{F}$
  - $\mathcal{F}$ is closed under countable unions: if $A_1, A_2, ..., A_n ... \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
- the probability measure $P : \mathcal{F} \to [0, 1]$ such that:
  - P is countably additive: if $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ is a countable collection of pairwise disjoint sets, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$
  - the measure of the entire sample space is equal to one: $P(\Omega) = 1$

The notion of a probability space is important, as Brownian motion is defined over a probability space.

Definition 3.2 (Filtration)**.** A filtration on $(\Omega,\ \mathcal{F},\ P)$ is a collection of measurable sets $\mathcal{F}_t : t \geq 0$ which satisfies $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ if $s < t$.

Intuitively, filtration represents all historical but not future information available about a stochastic process. This means that you cannot see into the future.

Definition 3.3 (Stopping Time)**.** A random variable $T : \Omega \mapsto [0, \infty]$ defined on a filtered probability space is called a stopping time with respect to the filtration $\mathcal{F}$ if the set $x \in \Omega : T(x) \leq t \in \mathcal{F}_t$ for all t.

For T to be a stopping time, it should be possible to decide whether or not $T(x) \leq t$ has occurred on the basis of the knowledge of $\mathcal{F}_t$.

Definition 3.4 (Markov Property)**.** Suppose that $\mathbf{X} = (X_t : t \in T)$ is a random process on the probability space $(\Omega, \mathcal{F}, P)$, where X_t is a random variable taking values in S for each $t \in T$. Then the random process $\mathbf{X}$ is said to satisfy the Markov property if for every $s, t \in T$ with $s \leq t$, and for every $H \in \mathcal{F}_s$ and $x \in S$, the conditional distribution of X_t given H and $X_s = x$ is the same as the conditional distribution of X_t just given $X_s = x$:

$$\mathbb{P}(X_t \in A | H, X_s = x) = \mathbb{P}(X_t \in A | X_s = x), A \subseteq S$$

That is, a stochastic process is said to have the Markov property if, given the present, the future does not depend on the past. One of the most famous Markov proccesses is a Markov chain. Brownian motion is also a Markov process.

Theorem 3.5 (Brownian motion and the Markov Property)**.** *Let $B_t : t \geq 0$ be a Brownian motion started at $x \in \mathbb{R}^d$. Fix $t > 0$, then the process $B_{t+s} - B_t : s \geq 0$ is a Brownian motion starting at the origin and independent of $B_t : 0 \leq t \leq s$. In other words, Brownian motion satisfies the Markov property.*

Proof. From the definition of Brownian motion, we know that Brownian motion satisfies the independent increments property (i.e. for any finite sequence of times $t_0 < t_1 < ... < t_n$, the distributions $B_{t_{i+1}} - B_{t_i}$ for $i = 1, ..., n$ are independent). Since the process

$$B_{t+s} - B_t = \sum_{j=1}^{s} B_{t+s} - B_{t+j-1}(s > 0)$$

where each term $B_{t+j} - B_{t+j-1}$ is independent, the given process is independent.

In probability theory, there is also the idea of a martingale. Essentially, a stochastic process is a martingale if the expected value of the next value is the sequence is equal to the present observed value, even given knowledge of all prior observed values. This means that the past events have nothing to do with predicting the future. More formally:

Definition 3.6 (Martingales)**.** In general, a stochastic process $B : T \times \Omega \mapsto S$ is a martingale with respect to a given filtration $\mathcal{F}^*$ and probability measure P if:

- $\mathcal{F}^*$ is a filtration of the probability space $(\Omega, \mathcal{F}, P)$
- For each t in T, each random variable B_t is measurable with respect to $\mathcal{F}_t$.
- $\mathbb{E}[B_t]$ is finite.
- For any $s, t \in T$ and $0 \leq s \leq t, \mathbb{E}[B_t|\mathcal{F}_s] = B_s$

Brownian motion is a martingale.

Another classic example of a martingale is Polya's urn, which contains a number of different coloured marbles. At each turn, a marble is randomly selected from the urn and replaced with several more of that same colour. The repeated selection of a ball from Polya's Urn is a martingale process, as the expected value of the next selection depends only on a current state of the urn.

There is also the notion of a local martingale, which satisfies the localized property of a martingale.

Definition 3.7 (Local Martingale)**.** A local martingale refers to an adapted stochastic process $X(t) : 0 \leq t \leq T$ which contains a sequence of stopping times T_n such that $X(min\{t, T_n\}) : t \geq 0$ is a martingale for every n.

4. Brownian Motion

Definition 4.1 (d-dimensional Brownian Motion)**.** A d-dimensional Brownian motion is a stochastic process $B_t : \Omega \to \mathbb{R}$ from the probability space (Ω, $\mathcal{F}$, P) to $\mathbb{R}^d$ such that the following properties hold:

- (Independent Increments) For any finite sequence of times $t_0 < t_1 < ... < t_n$, the distributions $B_{t_{i+1}} - B_{t_i}$ for $i = 1, ..., n$ are independent,

- For all $\omega \in \Omega$, the parametrization function $t \mapsto B_t(\omega)$ is continuous,
- (Stationary) For any pair $s, t \geq 0$, let $B_{s+t} - B_s \in A$,

$$P(B_{s+t} - B_s) = \int_A \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t} dx$$

Standard Brownian motion is the Brownian motion where $B_0(\omega) = 0$.

Definition 4.2 (Recurrence and Transience)**.** Brownian motion $B_t : t \geq 0$ is:

(1) transient if $lim_{t \to inf} |B_t| = \infty$
(2) point recurrent if for every $x \in \mathbb{R}^d$, there is an increasing sequence t_n such that $B_{t_n} = x$ for all $n \in \mathbb{N}$
(3) neighbourhood recurrent if for every $x \in \mathbb{R}^d$ and $\epsilon > 0$, there exists an increasing sequence t_n such that $B_{t_n} \in \mathbf{B}_\epsilon(x)$ for all $n \in \mathbb{N}$

Recurrence means that there is some probability greater than zero that the Brownian motion will go back to the place it came from. For example, if we are at the point one and if there is a probability greater than zero that we will return to one, then one is recurrent. Transience is the opposite of this.

Recurrence also gives us a criteria to define how Brownian motion moves in the $\mathbb{R}^d$ space. It's easy to figure out 1-dimensional Brownian motion has point recurrent from intuition. However, in higher dimensions (especially when $d \geq 3$), Brownian motion does not follow the same rule any more.

Theorem 4.3. *Brownian motion is:*

(1) *point recurrent in dimension $d = 1$,*
(2) *neighbourhood recurrent, but not point recurrent, in dimension $d = 2$ (also called as planar Brownian motion)*
(3) *transient in dimension when $d \geq 3$.*

5. Preliminaries for Proof of the Fundamental Theorem of Algebra

Theorem 5.1 (Fundamental Theorem of Algebra)**.** *If $f : \mathbb{C} \to \mathbb{C}$, $f(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ ($a_n \neq 0, a_0, ..., a_n \in \mathbb{C}$) is a polynomial of degree $n \geq 1$, then $f(\mathbb{C}) = \mathbb{C}$. Specifically, the equation $f(z) = 0$ has at least one root in $\mathbb{C}$.*

Note that Theorem5.1 covers real numbers, which simply have an imaginary part equal to 0, in addition to complex numbers. To prove Theorem 5.1, we will first define several terms and theorems. This proof follows the one given by Mihai N. Pascu, published in 2004 in a paper titled "A Probabilistic Proof of the Fundamental Theorem of Algebra."[1]

Definition 5.2 (P-Valued and P-Valent Functions)**.** For a given integer $p \geq 1$, a complex map $f : D \to \mathbb{C}$ is p-valued if for any $w \in \mathbb{C}$ there are at most p solutions in D to the equation $f(z) = w$, and there exists $w_0 \in \mathbb{C}$ for which the equation $f(z) = w_0$ has exactly p roots in D. A function for which the inverse is true (i.e., a value in the domain maps to at most p values in the codomain) is p-valent.

[1] `http://www.ams.org/journals/proc/2005-133-06/S0002-9939-04-07700-7/S0002-9939-04-07700-7.pdf`

Theorem 5.3. *Suppose that $f(z)$ is analytic at z_0, $f(z_0) = w_0$ and that $f(z) - w_0$ has a zero of order n at z_0. If $\epsilon > 0$ is sufficiently small, there exists a corresponding $\delta > 0$ such that for all w with $|w - w_0| < \delta$ the equation $f(z) = w$ has exactly n roots in the disk $|z - z_0| < \epsilon$.*

Theorem 5.3 roughly describes that the values around z_0 in the domain will map to values around w_0 in the codomain, provided several conditions hold.

Theorem 5.4 (Koebe's Quarter Theorem)**.** *The image of an injective analytic function $f : D \to \mathbb{C}$ from the unit disk D on to a subset of the complex plane contains the disk whose center is $f(0)$ and whose radius is $\frac{1}{4}|f'(0)|$.*

Koebe's Quarter Theorem is an important finding in complex analysis, and the constant of 1/4 cannot be improved. The following two theorems on Brownian motion will help us leverage the unique properties of this type of motion in order to construct a proof of the Fundamental Theorem of Algebra.

Theorem 5.5. *Given $\phi : [0, t] \to \mathbb{R}^d (d \geq 1)$ is continuous, B_t is a d-dimensional Brownian motion starting at $B_0 = \phi(0)$ and $\epsilon > 0$, then there exists $c > 0$ such that*

$$P^{\phi(0)}(\sup_{s \leq t} ||B_s - \phi(s)|| < \epsilon) > c, \tag{5.1}$$

where c depends only on t, ϵ, and the modulus of continuity of ϕ.

Theorem 5.5 states, roughly, that the probability that B_s approximates $\phi(s)$ is greater than 0. Since $\phi(s)$ is any continuous function defined on the domain $[0, t]$, the theorem suggests an important finding that Brownian motion at a certain point in time can be approximately modeled by a continuous function at that point in time.

Theorem 5.6. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire map and B_t a 2-dimensional Brownian motion starting at $B_0 = x$. Then $f(B_{\alpha_t})$ is a 2-dimensional Brownian motion starting at $f(B_0) = f(x)$, where*

$$\alpha_t = \inf\{s : A_s \geq t\} \tag{5.2}$$

and

$$A_t = \int_0^t |f'(B_s)|^2 ds. \tag{5.3}$$

Theorem 5.6 states, roughly, that an entire mapping of a Brownian motion can be expressed as a time shift of that motion. Before we can begin the proof, we will need the definition of an important topological property.

Definition 5.7. Given a domain D, a closed curve $\gamma \subset D$ is said to be homotopic to zero in D if the curve γ can be deformed continuously in D to a constant curve. It is known that homotopy is a topological property, meaning it is preserved under continuous mappings.

Throughout the proof, we will use $D(z, r)$ to indicate the open disk centered at $z \in \mathbb{C}$ of radius $r > 0$.

6. Proof of Fundamental Theorem of Algebra

Assume f never assumes $w_0 \in \mathbb{C}$, meaning $f(z) = w_0$ has no roots in $\mathbb{C}$. Since f is a non-constant polynomial of degree $n \geq 1$, there exists a $p \leq n$ such that f is p-valent.

By replacing $f(z)$ with

$$\tilde{f}(z) = \frac{f(z) - w_0}{w_1 - w_0}, \tag{6.1}$$

we can assume that $w_0 = 0$ and $w_1 = 1$. In other words, $f(z) = 0$ has no roots in $\mathbb{C}$, and $f(z) = 1$ has p roots in $\mathbb{C}$. Let $z_1, z_2, ..., z_m$ be distinct roots in $\mathbb{C}$ of $f(z) = 1$ with multiplicities $n_1, n_2, ..., n_m$. It follows then that $n_1 + n_2 + ... + n_m = p$.By the continuity of f and Koebe's quarter theorem, we can choose $\epsilon > 0$ small enough so that it satisfies

$$|f(z) - 1| < \frac{1}{4}\text{for all } z \in \bigcup_{i=1}^{m} D(z_i, \epsilon) \tag{6.2}$$

With Theorem 5.3, we can choose a smaller $\epsilon > 0$ so that there exist $\delta_1, \delta_2, ..., \delta_m > 0$ such that for $|w - 1| < \delta_i$, the equation $f(z) = w$ has exactly n_i roots in $|z - z_i| < \epsilon$, for all $i \in \{1, 2, ..., n\}$.Since $n_1 + n_2 + ... + n_m = p$ and f is p-valent, these roots are all the roots of the equation $f(z) = w$ in $\mathbb{C}$. Thus,

$$|f(z) - 1| < \delta \Rightarrow z \in \bigcup_{i=1}^{m} D(z_i, \epsilon) \tag{6.3}$$

where $\delta = min\{\delta_1, \delta_2, ...\delta_m\}$. Without loss of generality, we can also assume $\delta < \dfrac{1}{4}$.

Define a Brownian motion B_t in $\mathbb{C}$ starting at $B_0 = z_1$.Since f is entire, Theorem 5.6 implies that $W_t = f(B_{\alpha_t})$ is a Brownian motion starting at $f(B_0) = f(z_1) = 1$, where the time change α_t is the inverse of Equation 5.3.

Theorem 5.5 shows that there is a positive probability of

$$|W_t - e^{2\pi i t}| < \delta, \text{for all } t \in \big[0, m\big]. \tag{6.4}$$

We know $|W_j - e^{2\pi i j}| < \delta$, which means $|W_j - 1| < \delta$ and thus $|f(B_{\alpha,j}(\omega)) - 1| < \delta$. By equation 5.2, we know $B_{\alpha,j}(\omega) \in \bigcup\limits_{i=1}^{m} D(z_i, \epsilon)$

The box principle states that there exists $0 < j < k \leq m$ such that the Brownian motions $B_{\alpha,j}$ and $B_{\alpha,k}$ are elements of the disk $D(z_l, \epsilon)$ for some $1 \leq l \leq m$.

For the next step of the proof, we create a closed curve. We connect $[z_l, B_{\alpha,j}]$ and $[B_{\alpha,k}, z_l]$ to create a closed, continuous curve γ. Let Γ be the image of γ ($\Gamma = f(\gamma)$).

From the box principle, we know that $B_{\alpha,j}$ and $B_{\alpha,k}$ are elements of the disk $D(z_l, \epsilon)$. Since $z_l \in D(z_l, \epsilon)$, we know the whole path γ is a subset of $D(z_l, \epsilon)$ which is a subset of the union of the disks $\bigcup\limits_{i=1}^{m} D(z_i, \epsilon)$. Equation 5.2 states $|f(z) - 1| < \frac{1}{4}\ \forall\ z \in \bigcup\limits_{i=1}^{m} D(z_i, \epsilon)$. Thus, we know that $f([z_l, B_{\alpha,j}])$ and $f([B_{\alpha,k}, z_l])$ is a subset of $D(1, \frac{1}{4})$. This means that $W_t(\omega) = f(B_{\alpha,t}(\omega))$ lies in the the δ tube around the unit circle. Long story short, this shows that the index of the curve is nonzero.

A nonzero index implies that the path Γ is not homotopic to 0. This is a contradiction. We know γ is homotopic to 0 because closed continuous curves are homotopic to 0. This is because $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ and the fundamental group of $\mathbb{R}^2$ is 0. The image of a curve homotopic to 0 should also be homotopic to 0. However, this is not the case with the image of γ, Γ.

Our original assumptions stated that f did not cover 0 and the above contradiction shows that this is not possible. Thus, f covers every value of $\mathbb{C}$ or $f(\mathbb{C}) = \mathbb{C}$.

7. Preliminaries for Louville's Theorem with Brownian Motion

Recall: Brownian motion is neighborhood recurrent (but not point recurrent) in $d = 2$

Remark: Because Brownian motion is neighborhood recurrent in $d = 2$, we know that the path of the planar Brownian motion is dense in the plane. What this means, informally, is that Brownian motion goes to every point in the plane or goes arbitrarily close to it.

Theorem 7.1 (Optional Stopping Theorem)**.** *Suppose X_t is a continuous martingale and $0 \leq S \leq T$ are stopping times. If the process $X(min\{t, T\})$ is dominated by an integrable random variable Y, then $\mathbf{E}[X_T|\mathcal{F}_S] = X_S$.*

Suppose we wanted to talk about games then a martingale is just a way to describe a fair game. The money a player is expected to end with is simply the money they have right now. However, if we were to end the game at certain stopping times then would the person still have the expected amount of money? In some situations yes and in others no. This theorem just explains what conditions must be met in order for the expected value of the random variable at stopping time to equal the initial value.

Recall that while Brownian motion is continuous everywhere, it is can be differentiated nowhere. So what does calculus in such a place even mean? To understand and operate on a system we have to make something different. This is why stochastic calculus is so important since it can be used to operate on stochastic systems. Specifically we will be using Ito's process and Ito's formula help us.

Many times when studying Brownian motions, we want to estimate the difference of functions like $f(t, B_t)$ over time differences that infinitesimally small (let us assume that f is a smooth function). Notice that this function only depends on the the second variable, so there is only an implicit dependence on time.

Let us consider the Taylor expansion of f

$$f(x + \Delta x) - f(x) = (\Delta x)f'(x) + \frac{(\Delta x)^2}{2}f''(x) + \dots$$

Since we are using B_t as the input, for $x = B_t$, we have

$$\Delta f = (\Delta B_t)f'(B_t) + \frac{(\Delta B_t)^2}{2}f''(B_t) + \dots$$

Consider the term $(\Delta B_t)^2$. Since we know that $\mathbf{E}[(\Delta B_t)^2] = \Delta t$, we see that the second term is no longer negligible (this is called quadratic variation). The equation becomes

$$\Delta f = (\Delta B_t)f'(B_t) + \frac{\Delta t}{2}f''(x) + \dots$$

and in terms of infinitesimals, we have

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t) * dt$$

which is known as Ito's lemma. More generally we have the formal statement of Ito's lemma.

Theorem 7.2 (Ito's Lemma)**.** *Let $f(t,x)$ be a smooth function of two variables, and let X_t be a stochastic process satisfying $dX_t = \mu_t dt + \sigma_t dB_t$ for a Brownian motion B_t. Then*

$$df(t,X_t) = \big(\frac{\partial f}{\partial t}dt + \mu_t\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2\frac{\partial^2 f}{\partial x^2}\big)dt + \frac{\partial f}{\partial x}dB_t$$

Because of Ito's lemma, we can get the following theorem, which is necessary for our proof of Liouville.

Theorem 7.3. *Let $D \subset \mathbb{R}^d$ be a connected open set and $f : D \to \mathbb{R}$ be harmonic on D. Let B_t with $0 \le t \le T$ be Brownian motion starts inside D and stops at T, then process $f(B_t) : 0 \le t \le T$ is a local martingale.*

All that this is saying is that if we fulfill such conditions, and the Brownian motion takes a specific path then the function during a the interval from 0 to T is a local martingale.

Theorem 7.4. *(Levy's Theorem) Suppose that both M and $(M_t^2 - t)_{t\ge 0}$ are local martingales. Assume $M_0 = 0$. Then M is a Brownian motion with respect to $(\mathcal{F}_t)$.*

Proof. Let $f(x) = e^{ivx}$ where $v \in \mathbf{R}$. Since $f \in [0,\infty)^2$, we have by Ito's Lemma

$$f(M_t) = f(0) + \int_0^t f'(M_s)dM_s + \frac{1}{2}\int_0^t f''(M_s)ds$$

where $M_t^f := \int_0^t f'(M_s)dM_s$ is a local martingale. If f' and f'' are bounded, then M_t^f is a martingale. If we take the expectations of both sides we get

$$\mathbf{E}[f(M_t)] = f(0) + \frac{1}{2}\int_0^t \mathbf{E}[f''(M_s)]ds$$

Let $g(t) = \mathbf{E}[f(M_t)]$, then we get

$$g(t) = 1 - \frac{v^2}{2}\int_0^t g(s)ds$$

This shows that we can get g(t) as the solution to the differential equation that satisfies the following conditions:

$$g'(t) = -\frac{v^2}{2}g(t) \text{ and } g(0) = 1$$

As a first order differential equation, the unique solution to g is

$$g(t) = e^{-\frac{tv^2}{2}}.$$

As a result, we have

$$\mathbf{E}[e^{ivM_t}] = e^{-\frac{tv^2}{2}},$$

displaying that $M_t \backsim \mathcal{N}(0,t)$ where t is the variance. Let $s > 0$ and $A \in \mathcal{F}_s$ with $P(A) > 0$. Let $P^*(B) = P(B|A)$, $\mathcal{F}_t^* = \mathcal{F}_{t+s}$ and $M_t^* = M_{t+s} - M_t$ for $t \ge 0$. Then with respect to $\mathcal{F}_t^*$ over probability space $(\Omega, \mathcal{F}, P^*)$, $(M_t^*)_{t\ge 0}$ is continuous local martingale with $M_0^* = 0$ such that $[M_t^*]^2 - t$ is also a local martingale. Then from what we have already proven,

$$\mathbf{E}[e^{ivM_t^*}] = e^{-\frac{tv^2}{2}}$$

Substitute $M_t^* = M_{t+s} - M_t$ and let A vary in $\mathcal{F}_s$

$$\mathbf{E}[e^{iv(M_{t+s}-M_t)}][\mathcal{F}_s] = e^{-\frac{tv^2}{2}}$$

which shows that $M_{t+s} - M_t$ is independent of $\mathcal{F}_s$ and is normally distributed, making it Brownian motion.

Theorem 7.5 (Dubins-Schwartz Theorem)**.** *Let $(M_t)_{t\geq 0}$ is a continuous local martingale such that M_0=0 and $\langle M\rangle_\infty = +\infty$. There exists a Brownian motion $(B_t)_{t\geq 0}$, such that for every $t \geq 0$, $(M_t) = B_{\langle M\rangle_t}$.*

Proof. Let $C_t = \inf\{s \geq 0, (M_t)_s < t\}$. $(C_t)_{t\geq 0}$ is a right continuous and increasing process such that for every $t \geq 0$, C_t is a finite stopping time of the filtration $(\mathcal{F}_t)_{t\geq 0}$ and M is obviously constant on each interval $[C_{t-}, C_t]$. $(M_{C_t})_{t\geq 0}$ is a local martingale whose quadratic variation is equal to t. From Levy's theorem, $(B_t)_{t\geq 0}$ is a Brownian motion.

8. Proof of Liouville's Theorem with Brownian Motion

Theorem 8.1 (Liouville's Theorem)**.** *Suppose f is complex valued function that is entire and bounded, then it is constant.*

Proof. Suppose f is an entire function but not constant. Then according to 7.3 when given a Brownian motion B_t, $f(B_t)$ will be a local martingale. By 7.5, $f(B_t)$ is also a Brownian motion. Recall that planar Brownian motion is neighborhood recurrent, so f is dense and therefore f is not bounded, contradicting our assumption. Therefore, the entire function must be constant.