Defining the Riemann Zeta Function

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On the real line with $x > 1$, the Riemann zeta function can be defined as

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} du, \tag{1}$$

where $\Gamma(x)$ is the gamma function.[1] If x is an integer n, then we have the identity

$$\begin{aligned} \frac{u^{n-1}}{e^u - 1} &= \frac{e^{-1} u^{n-1}}{1 - e^{-u}} \\ &= e^{-1} u^{n-1} \sum_{k=0}^{\infty} e^{-ku} \\ &= \sum_{k=1}^{\infty} e^{-ku} u^{n-1}, \end{aligned}$$

so

$$\int_0^\infty \frac{u^{n-1}}{e^u - 1} du = \sum_{k=1}^{\infty} \int_0^\infty e^{-ku} u^{n-1} du.$$

Let $y = ku$, so that $dy = kdu$. Substituting into equation 1, we obtain

$$\begin{aligned} \zeta(n) &= \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \int_0^\infty e^{-ku} u^{n-1} du \\ &= \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \int_0^\infty e^{-y} \left(\frac{y}{k}\right)^{n-1} \frac{dy}{k} \\ &= \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \frac{1}{k^n} \int_0^\infty e^{-y} y^{n-1} dy. \end{aligned}$$

The integral on the right-hand side gives $\Gamma(n)$, which cancels out the $\frac{1}{\Gamma(n)}$ and gives the most common form[2] of the Riemann zeta function,

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

[1] The gamma function is defined to be an extension of the factorial to both complex and real arguments. It is defined as $\Gamma(n) = (n-1)!$.

[2] This is sometimes known as a p-series.