Differential Posets

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Abstract

We introduce differential posets, a class of partially ordered sets which have two operators, namely U (up-map) and D (down-map). We explore the theorems and conjectures associated with these posets as well as a brief overview of their applications to important algorithms such as the Robinson-Schented-Knuth correspondence (RSK algorithm).

1 Introduction

Partially ordered sets or posets are sets that are equipped with a binary relation which describes how elements are ordered. We start by defining the definition of a *graded* poset.

Definition 1. *A poset P is said to be* graded *if there exists a rank function $f : P \to N$ such that:*

1) For $a, b \in P$, if $a < b$, $f(a) < f(b)$.
2) For $a, b \in P$, if a covers b, then $f(a) = f(b) + 1$

Note that an element A "covers" another element B in a poset P if A is "greater" than B in terms of the partial ordering defined and there is no element $C \in P$ such that $B < C < A$. In other words, the interval $[A, B]$ contains only the elements A and B.

We now define the main theory of this paper, namely the class of posets known as differential posets.

Definition 2. *A poset P is an r-differential poset if:*

1) P is locally finite and graded and has a minimal element $(\hat{0})$.
2) For any two distinct elements a, b of P, both a and b are covered by the same amount of elements as the number of elements that they cover.

3) If an element $a \in P$ covers b elements, then it is covered by $b + r$ elements.

In addition, it is important to define what is meant by the "up" and "down" maps or U and D respectively:

Definition 3. *For a differential poset P, U is the "up map", which means that $U(x) = \{y \in P \mid y \text{ covers } x\}$. Similarly, D is the "down map", which means that $D(x) = \{y \in P \mid y \text{ is covered by } x\}$.*

Finally, we define important examples of differential posets, namely Young's Lattice and how it is formed.

Young Diagrams are labelings of integer partitions in non-decreasing order. Imposing an ordering upon these diagrams in which a diagram is said to be "less" than another if it is contained in another diagram, forming Young's Lattice, which is a poset based on integer partitions. By filling in the elements of Young Diagrams, you create Young Tableaux.

In terms of the ordering of the poset defined by Young's Lattice, it is such that if we have two Young Tableaux A and B with their partitions being $a_1 + a_2 + ... + a_k = A$ and $b_1 + b_2 + ... + b_j = B$, then $A < B$ if $a_i \leq b_i$ for all i from 1 to $min(j, k)$. An example of Young's Lattice is shown in Figure 1.

2 Important Examples and Theorems

Theorem 2.1. *Young's Lattice is a 1-differential poset.*

Proof: We show that Young's Lattice satisfies all of the three required axioms of being an r-differential poset, with $r = 1$. Denote the poset associated with Young's Lattice P.

Firstly, P has a $\hat{0}$ which is namely $\emptyset$. P is locally finite because for every pair of Young Tableaux (A, B), there are a finite number of other tableaux in the interval $[A, B]$. P is graded because we can let the partitions of an element $n \in P$ have rank n, thus satisfying the requirements previously mentioned about graded posets.

Secondly, for any two distinct elements $a, b \in P$, we have that both a and b are covered by the union of the Young's Lattices of a and b. In the same way, a and b cover the intersection of the Young's Lattices of a and b. Thus, a and b are covered by 1 element and cover 1 element, proving the second axiom.

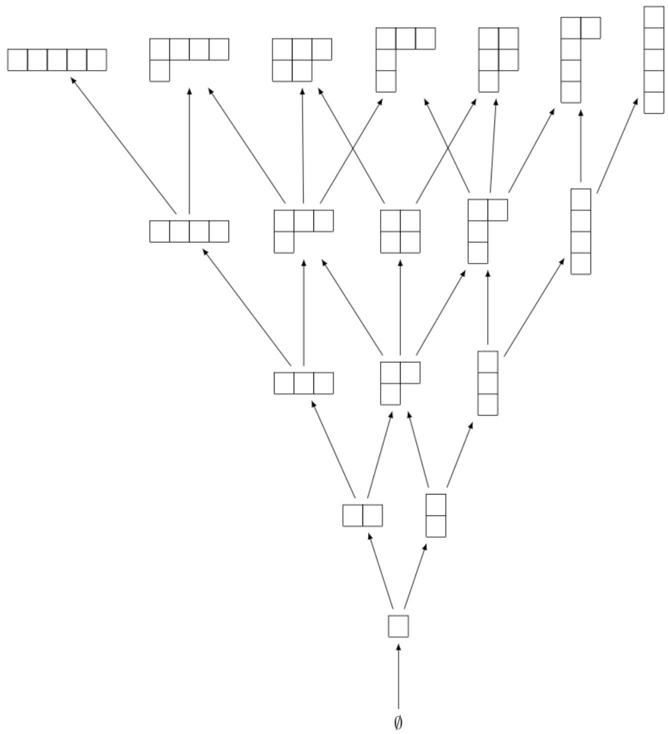

Figure 1: Example Diagram of Young's Lattice [1]. Note that an arrow from a Young Tableaux A to another Young Tableaux above it B is such that $A < B$ under the aforementioned ordering relation.

Finally, if $a \in P$ covers b elements, then we have that b equals the number of distinct parts in the partition as we could remove one from any one of the parts in the partition to make a partition covered by a. In the same way, a will then be covered by $b+1$ elements as we could either add 1 to a part of the partition or create a separate part of size 1 in the partition.

$\square$

Corollary 2.1.1. *Suppose we have a poset P such that for $a, b \in P$ and $a \neq b$, there are either no elements or 1 element which covers both a and b.*

This follows from the fact that the element which covers both a and b is their union (or no element).

Theorem 2.2. *If P is r-differential and Q is s-differential, then $P \times Q$ is $(r+s)$-differential.*

Proof: It is quite trivial to see that the theorem holds for axioms 1 and 2 by theorem 2.1 as it is just an extension of single posets to ordered pairs of posets. For axiom 3, note that if an element $(a, b) \in P \times Q$ covers c elements, then if we let S_P denote the set of all posets that a covers in P and S_Q denote the set of all posets that b covers in Q. Then, note that (a, b) covers $c + |S_P| + |S_Q| = c + r + s$ elements.

□

Remark. *Just to clarify, note that $P \times Q$ denotes the Cartesian product of the posets P and Q and are thus under the partial ordering $(a, b) < (c, d)$ if $a <_1 b$ and $c <_2 d$.*

The main idea behind differential posets is in the use of the "up" and "down" maps. We further clarify the meaning of the U and D maps:

$$Ux = \sum_{a<x} a$$

$$Dx = \sum_{a>x} a$$

Theorem 2.3. *Given that a poset P satisfies the first condition (outlined in Definition 1), then P is r-differential iff:*

$$DU - UD = rI$$

Proof: Proven in 3.21.3 in Stanley's *Enumerative Combinatorics* [2].

3 Applications - RSK Algorithm

The RSK algorithm or Robinson-Schensted-Knuth correspondence has a wide variety of applications and has interesting combinatorial properties. I found its symmetry to be especially interesting, and before we dive into some of the theory of the RSK algorithm, we need to define what Semi-Standard Young Tableaux (SSYT) and Standard Young Tableaux (SYT) are.

Semi-Standard Young Tableaux are Young Tableaux where there is elements increase in a weak order along the rows while the column sizes still strictly decrease. Standard Young Tableaux are a specific class of SSYTs for which each integer from 1 to n (where n is the number being partitioned) appears exactly once. An example of a semi-standard Young Tableaux is shown below.

1	1	1	2	3
3	4	5		
4	5			
6				

The RSK algorithm is a bijection between pairs of Standard Young Tableaux (SYT) (A, B) and elements $\phi \in S_n$, where the size of the SYT equals n. The basic idea behind proving this is that since each element from 1 to n appears in Standard Young Tableaux exactly once, then if A is the matrix associated with the RSK algorithm for the pair (A, B), then the sum of the rows and the sum of the colums equals 1, giving us that there is exactly one 1 in each row and one 1 in each column, while the rest of the matrix consists of just 0's. Since our process is completely invertible (where we go from the matrix to the pair of tableaux), we have a bijection.

References

[1] Chen Avin, Avi Cohen, Pierre Fraigniaud, Zvi Lotker, and David Peleg. Preferential attachment as a unique equilibrium. In *Proceedings of the 2018 World Wide Web Conference*, WWW '18, page 559–568, Republic and Canton of Geneva, CHE, 2018. International World Wide Web Conferences Steering Committee.

[2] Richard P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, USA, 2nd edition, 2011.