Convex Polytopes

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Abstract

In this paper, we aim to explore some of the properties of complex polytopes, specifically, the intersection of convex sets, lattice points in sets, as well as combinatorial interpretations of the number faces and edges in special types of polytopes. Most of the definitions in the introduction are taken from Ziegler's "Lectures on Polytopes" [13]. In addition, the theorems that relate to convex sets are taken from [1] and [4]. The content in the section on associahedrons is largely based on Loday's paper [2], with a few of my own modifications, such as the representation of vertices.

1 Introduction and Examples

We begin by defining a *convex set*, which is essential in defining a polytope.

Definition 1.1 (Convex Set)**.** A point set $K \subseteq \mathbb{R}^n$ is *convex* if for any points $\boldsymbol{x}, \boldsymbol{y} \in K$ it also contains the straight line segment between them.

In Figure 1, for example, the set on the left is convex because all line segments between two points in the set are also in the set, while the set on the right is not convex because not all such line segments are in the set.

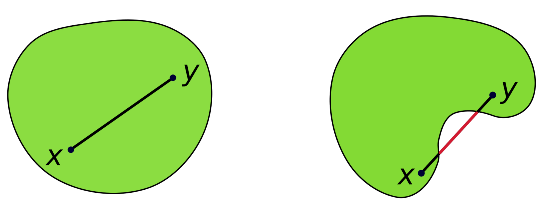


Figure 1: Convex set on left; non-convex set on right [7]

One important property of a convex set is that every intersection of convex sets is also convex. With this in mind, we can construct a *convex hull*.

Definition 1.2 (Convex Hull)**.** A convex set is a *convex hull* of $K \in \mathbb{R}^n$ if it is the intersection of all convex sets containing K:

$$\mathrm{Conv}(K) := \bigcap \left\{K' \in \mathbb{R}^d : K \subseteq K', K' \text{ convex}\right\}.$$

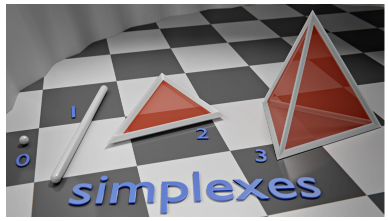


Figure 2: Simplices in multiple dimensions [10]

In addition, we have an alternate definition of the convex hull in terms of *convex combinations*. A convex combination is simply a linear combination of points in a set such that the coefficients of the points are nonnegative and sum to 1.

Definition 1.3 (Convex Hull)**.** A convex hull of some subset is the set of all convex combinations of points in the subset.

Roughly speaking, the convex hull can be thought of as the "smallest" convex set containing K. Now that we have these basic definitions, let's introduce polytopes:

Definition 1.4 ($\mathcal{V}$-polytope)**.** A $\mathcal{V}$*-polytope* is a convex hull of a finite set of points in $\mathbb{R}^n$.

In addition, we have the definition for an $\mathcal{H}$*-polytope*:

Definition 1.5 ($\mathcal{H}$-polytope)**.** An $\mathcal{H}$*-polytope* is a bounded intersection of finitely many closed halfspaces in $\mathbb{R}^n$.

A *polytope* then, is a point set $P \subseteq \mathbb{R}^n$ which is either a $\mathcal{V}$-polytope or $\mathcal{H}$-polytope. In addition, a *convex polytope* is a polytope that is also a convex set in $\mathbb{R}^n$. We will refer to convex polytopes simply as polytopes. A d*-polytope* is a polytope of dimension d in some $\mathbb{R}^n$ with $n \geq d$. A polytope contains elements of different dimensions less than d that are referred to as *faces*. These faces include the empty set, points, lines, and even the polytope P. One-dimensional faces are defined as points, two-dimensional faces are defined as lines, and $(d-1)$-dimensional faces are defined as *facets*.

Example. One of the most fundamental polytopes that exists is known as the *simplex*. In one dimension, the simplex is a line. In two dimensions, the simplex is simply a triangle. In three dimensions, the simplex is a tetrahedron, and in four dimensions it is known as a 5-cell. In general, a k-simplex is a k-dimensional polytope which is the convex hull of $k+1$ vertices (see figure 2).

Example. The *hypercube* is another common type of polytope. It is a generalization of a square or a cube to an n-dimensional figure. In four dimensions, the 4-cube is known as a tesseract. Strictly speaking, the hypercube is a closed, compact, convex figure whose 1-skeleton is made up of opposite and parallel groups of line segments that are perpendicular to each other and have the same length.

In this paper, we will begin by focusing our attention to some of the important properties and results of convex sets. Then, we will discuss two specific polytopes known as the *permothedron* and *associahedron*.

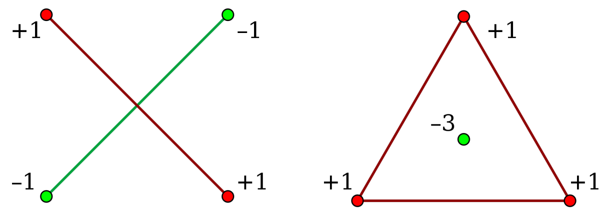


Figure 3: Disjoint subsets whose convex hulls share a common point [9]

2 Convex Sets

Since polytopes are convex sets, let's introduce a few of the important results that come from convex sets. We will begin by introducing *Radon's Theorem*, and a related result *Helly's Theorem*. We will then shift focus toward convex sets in lattices and important results such as *Blichfeldt's Theorem* and *Minkowski's Theorem*.

2.1 Intersections of Sets

In order to prove Helly's Theorem, let's first establish Radon's Theorem.

Theorem 2.1 (Radon's Theorem [1]). *Each set of $n+2$ or more points in $\mathbb{R}^n$ can be expressed as the union of two disjoint subsets whose convex hulls have a common point.*

Example. To visualize this, consider $X \subset \mathbb{R}^2$. Radon's Theorem says that if X contains 4 points, then there exists an intersection either between the convex hull of 3 of the points and the 1 left over or between 2 line segments. In other words, we can either form a triangle out of 3 of the points such that the remaining point is inside it, or we can form two intersecting line segments, as shown in Figure 3.

Proof. Let P be a set of $m \geq n+2$ points p_i in $\mathbb{R}^n$ with $1 \leq i \leq m$. Then, since $m \geq n+2$, the points in X are affine dependent, and there exists some $(a_0, a_1, \ldots, a_m)$ such that

$$\sum_{i=0}^{m} a_i p_i = 0, \sum_{i=0}^{m} a_i = 0.$$

Now, split the indices of the points into two disjoint subsets I_1 and I_2 such that all of the non-negative a_i go to I_1 and all of the negative a_i go to I_2. In other words,

$$I_1 \coloneqq \{1, \ldots, m \mid a_i \geq 0\}, I_2 \coloneqq \{1, \ldots, m \mid a_i < 0\}.$$

In addition, let $A = \sum_{i \in I_1} a_i$. Then,

$$\sum_{i \in I_1} a_i + \sum_{i \in I_2} a_i = 0$$
$$A = \sum_{i \in I_1} a_i = -\sum_{i \in I_2} a_i.$$

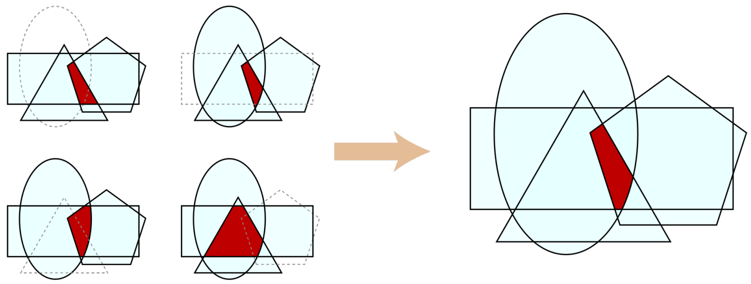


Figure 4: Helly's Theorem [8]

Lastly, let $x = \sum_{i\in I_1} \frac{a_i}{A} p_i$. This leaves

$$\sum_{i\in I_1} a_i p_i + \sum_{i\in I_2} a_i p_i = 0$$
$$\sum_{i\in I_1} \frac{a_i}{A} p_i + \sum_{i\in I_2} \frac{a_i}{A} p_i = 0$$
$$x = \sum_{i\in I_1} \frac{a_i}{A} p_i = \sum_{i\in I_2} \frac{-a_i}{A} p_i.$$

From the last equation, we see that x is a complex combination of the points in I_1 and a complex combination of the points in I_2 since

$$\sum_{i\in I_1} \frac{a_i}{A} = \sum_{i\in I_2} \frac{-a_i}{A} = 1.$$

Therefore, there is some point x that exists inside the convex hulls of two disjoint subsets of P. □

One of the most important applications of Radon's Theorem is its usefulness in proving Helly's Theorem.

Theorem 2.2 (Helly's Theorem [1])**.** *Suppose X is a collection of $m \geq n+1$ convex sets in $\mathbb{R}^n$, and X is finite or each set in X is* compact. *Then if each $n+1$ members of X have a common point, there is a common point to all members of X.*

Proof. Let S_i denote each convex set in X with $1 \leq i \leq m$. We wish to show that

$$\bigcap_{i=1}^{m} S_i \neq \emptyset.$$

In order to prove this result, we will form an inductive argument. To start, note that the base case $m = n+1$ is trivial. Now, assume that the condition holds when $|X| = m \geq n+1$. The inductive step will go as follows:

Since the condition holds when $|X| = m \geq n+1$, there exists a point p_j with $1 \leq j \leq m+1$ in the intersection of all S_i except for possibly one, say S_j. Let $P = \{p_1, p_2, \ldots, p_{m+1}\}$. Then,

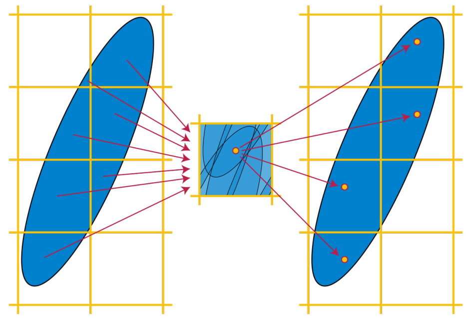


Figure 5: Translating each S_i to the origin [6]

2.1 tells us that we can split P into P_1 and P_2 such that there is a point p in the intersection of the convex hulls of P_1 and P_2. We claim

$$p \in \bigcap_{i=1}^{m+1} S_i.$$

We wish to show p exists in every S_i. To do this, note that all points in P are in S_i except for possibly p_i, since all other p_j with $j \neq i$ are in the intersection of some combination of Ss including S_i. Now, consider the case where $p_i \in P_1$. Then, $p_i \notin P_2$ and all of the points in P_2 are in S_i. Since S_i is convex,

$$\mathrm{Conv}(P_2) \subset S_i.$$

Therefore, $p \in S_i$ for all $1 \leq i \leq m+1$. Similarly, if $p_i \in P_2$ instead, we have $p \in S_i$ for all $1 \leq i \leq m+1$ as well. □

2.2 Lattice Points in Sets

Now that we have discussed intersections of convex sets, let's shift focus toward counting lattice points in convex sets. One of the most important results in this area is Minkowski's Theorem. In order to prove it though, let's establish a related result—Blichfeldt's Theorem.

Theorem 2.3 (Blichfeldt's Theorem [4])**.** *Let S be a bounded set of points in $\mathbb{R}^2$ whose area A exceeds 1. Then there exist two points x_1 and x_2 such that $x_1 - x_2$ has integral coordinates.*

While we will consider the $\mathbb{R}^2$ case, Blichfeldt's Theorem can be generalized to n dimensions as well.

Proof. We begin by covering S with unit squares that are formed from the integer lattice $\mathbb{Z}^2$. Let $S_1, S_2, \ldots S_k$ denote the intersections between the unit squares and S. Now, translate each S_i with $1 \leq i \leq k$ to the origin. This can be thought of as stacking all of the tiles with a piece of S on top of each other so that they fit inside a unit square. Then, since $A > 1$, there exists some point z in the intersection of the translated copies of S_a and S_b where $a \neq b$ and $1 \leq a, b \leq k$. Let l_1 denote the lattice point corresponding to S_a and let l_2 denote the lattice point corresponding to S_b. Then, the points $x_1 = l_1 + z$ and $x_2 = l_2 + z$ both exist in S. In addition,

$$x_1 - x_2 = l_1 + z - (l_2 + z) = l_1 - l_2,$$

which is a point on $\mathbb{Z}^2$. □

Now, let's apply 2.3 in order to prove an important corollary.

Corollary 2.3.1 (Minkowski's Theorem)**.** *If M is a symmetrical bounded convex set with an area greater than* 4*, then it contains at least one lattice point other than the origin.*

Proof. Let $\hat{M} = \frac{1}{2}M$, so that all of the points in M are shrunk by a factor of 2, and let $A(x)$ denote the area of a convex set x. Then,

$$A(\hat{M}) = \left(\frac{1}{2}\right)^2 A(M) > 1.$$

Now, Blichfeldt's Theorem tells us that there exist x_1 and x_2 in $\hat{M}$ such that $x_1 - x_2$ has integral coordinates. Since x_1 and x_2 exist in $\hat{M}$, the points $2x_1$ and $2x_2$ exist in M. In addition, since M is symmetric and convex $-2x_2$ exists in M and so does the midpoint of $2x_1$ and $-2x_2$. The midpoint of $2x_1$ and $-2x_2$ is simply

$$\frac{2x_1 - 2x_2}{2} = x_1 - x_2,$$

which has integral coordinates. Therefore, there is a point with integral coordinates in M. □

Minkowski's Theorem has a number of applications to number theory and other areas in the theory of convex sets. Minkowski's Theorem can be used along with Euler's Formula to prove Pick's Theorem, for example, which provides a formula for the area of polygons with lattice point vertices.

3 Permutohedron and Associahedron

We have layed the foundation for describing polytopes by considering important results from convex sets. Now, let's shift focus to *polyhedral combinatorics.*

3.1 Permutohedron

When discussing polyhedral combinatorics, the *permutohedron* is an interesting polytope to consider.

Definition 3.1 (Permutohedron)**.** The *permutohedron* is defined $\Pi_{n-1} \subseteq \mathbb{R}^n$ is defined as the convex hull of all vectors that are obtained by permuting the coordinates of the vector $\begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$.

From this definition, we see that the permutohedron is the polytope whose vertices are permutations of [n] and whose edges connect transpositions of permutations from two vertices.

Example. Let's consider the permutohedron of order 4, a truncated octahedron. Then there are $4! = 24$ vertices corresponding to the 24 permutations of $\{1, 2, 3, 4\}$. In addition, to go from one vertex to another connected by an edge, simply swap the order of two numbers in the permutation (see Figure 6). The facets of the permutohedron correspond to the 2-dimensional faces of the truncated octahedron.

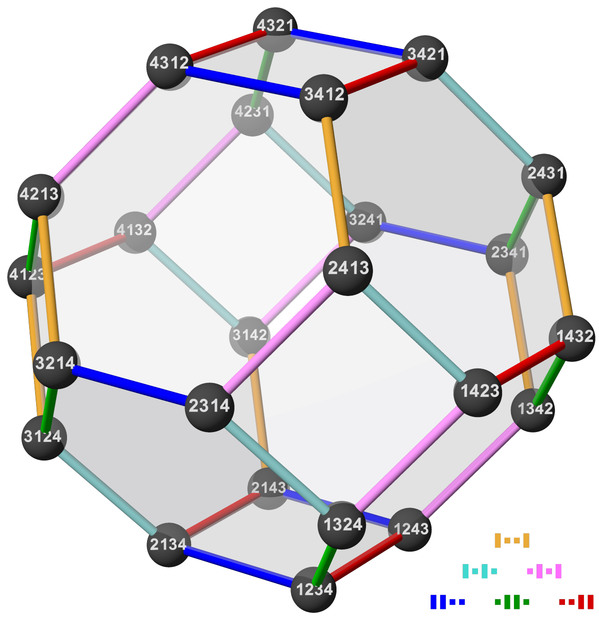


Figure 6: Permutohedron of order 4 [12]

Immediately, we can see that the number of vertices is equivalent to $n!$, since this is simply the number of permutations of [n]. In addition, we can clearly see that the number of edges is equivalent to $\frac{(n-1)n!}{2}$ since each vertex has $n-1$ edges attached to it.

At this point, we may ask whether there is a way to describe the number of $(n-k)$-faces of a permutohedron. It turns out that there is indeed a nice way of counting such faces. However, before we accomplish this, let's count the number of facets of the permutohedron.

Lemma 3.1. *Let Π_{n-1} be a permutohedron in $\mathbb{R}^n$. Then the number of facets of Π_{n-1} is equal to $2^n - 2$.*

Proof. Let S denote a nonempty proper subset of [n]. Then, construct a facet from the vertices whose permutations contain the smallest elements at the positions indicated by S. For example, if $n = 4$ and $S = \{1, 3\}$, construct a facet from vertices $(1, 3, 2, 4)$, $(1, 4, 2, 3)$, $(2, 3, 1, 4)$, and $(2, 4, 1, 3)$. Then, the facets are in bijection with the subsets S, so the number of facets is equal to the number of total subsets minus the empty subset and [n]. This is simply $2^n - 2$. □

Now, let's use this result to describe the number of $(n-k)$-faces.

Theorem 3.2. *The number of faces of dimension $n-k$ in Π_{n-1} is equal to $k!S(n,k)$, where $S(n,k)$ represents the Stirling numbers of the second kind.*

Proof. To start, note that the permutohedron is a simple polytope, so the intersection of k facets is a $(n-k)$-face. Now, consider two intersecting facets F_1 and F_2 denoted by subsets S_1 and S_2, respectively. In addition, let $i = |S_1| < j = |S_2|$. Then, F_1 has vertices whose permutations have $1, 2, \ldots, i$ in the positions stored in S_1, and F_2 has vertices whose permutations have $1, 2, \ldots, i, \ldots, j$ in the positions marked by S_2. Since F_1 and F_2 intersect, they must share at least one common vertex, meaning $S_1 \subset S_2$. In addition, since $S_1 = S_2 \implies F_1 = F_2$, we must have

$$\emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq [n].$$

Similarly, if we intersect $k-1$ facets, constructed from proper subsets $S_1, S_2, \ldots, S_{k-1}$ of $[n]$, we must have

$$\emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq [n].$$

So, the total number of $(n-k)$-faces is equivalent to the number of sequences of nonempty nested proper subsets $S_1, S_2, \ldots, S_{k-1}$. We claim that these sequences of subsets are in bijection with the partitions of $[n]$ into k nonempty subsets counting order. To show this, partition $[n]$ into $T_1, T_2, \ldots, T_k$. Then, the sequence of subsets can be constructed as

$$\emptyset \subsetneq T_1 \subsetneq T_1 \cup T_2 \subsetneq \cdots \subsetneq \bigcup_{i=1}^{k-1} T_i \subsetneq [n].$$

In addition, we can also construct a partition of $[n]$ into k subsets from a sequence of $k-1$ subsets denoting facets. Therefore, since there is indeed a bijection, the total number of $(n-k)$-faces is equal to the number of partitions of $[n]$ into k nonempty subsets counting order, that is, $k!S(n,k)$. [3] $\square$

3.2 Associahedron

In addition to the permutohedron, there exists another similar polytope whose vertices have an important combinatorial interpretation. We call this polytope the *associahedron*. Before we define the associahedron, let's introduce some notation:

Let Y_n be the set of parenthesizations of a word with $n+1$ letters.

Example. We have,

$$Y_1 = \{(x_0 x_1)\},$$
$$Y_2 = \{((x_0 x_1)x_2), (x_0(x_1 x_2))\},$$
$$Y_3 = \{((x_0 x_1)(x_2 x_3)), (x_0((x_1 x_2)x_3)), (x_0(x_1(x_2 x_3))), (((x_0 x_1)x_2)x_3), ((x_0(x_1 x_2))x_3)\}.$$

Now, define $t \vee s$ to be a parenthesization that takes t as a left element and that takes s as a right element. In other words,

$$t \vee s = (t, s).$$

We can associate any t with an integral vertex $M(t)$ as follows. Let $x_0, x_1, x_2, \ldots, x_n$ be the elements in t from left to right. Then, place some divider d_i between x_{i-1} and the first group or element to the right of x_{i-1}. Let a_i be the number of x_is to the left of d_i inside the parenthesization containing d_i, and let b_i be the number of x_is to the right of d_i inside the parenthesization. Define $M(t)$ as

$$M(t) \coloneqq (a_1 b_1, a_2, b_2, \ldots, a_n b_n).$$

Example. Suppose we have $t = (((x_0, x_1)(x_2 x_3))x_4)$. Then, we can insert the d_is to get $(((x_0 d_1 x_1)d_2(x_2 d_3 x_3))d_4 x_4)$. This leaves

$$M(t) = (1 \times 1, 2 \times 2, 1 \times 1, 4 \times 1) = (1, 4, 1, 4).$$

Lastly, define H_n to be a hyperplane in $\mathbb{R}^n$ whose equation is

$$x_1 + x_2 + \cdots + x_n = \frac{n(n+1)}{2}.$$

Lemma 3.3. *The point $M(t)$ belongs to hyperplane H_n for all $t \in Y_n$.*

Proof. We construct a proof by induction. The $n = 1$ case is satisfied because (1) belongs to $x_1 = 1$. Now, break t up into $t_1 \vee t_2$ where $t_1 \in Y_p$ and $t_2 \in Y_1$ and assume that the condition holds when $n = p$ and $n = q$. We have

$$M(t) = (M(t_1), (p+1)(q+1), M(t_2)).$$

Let z_i denote the ith coordinate in $M(t)$. Then, since $n = p + q + 1$,

$$\sum_{i=1}^{n} z_i = \frac{p(p+1)}{2} + (p+1)(q+1) + \frac{q(q+1)}{2} = \frac{n(n+1)}{2}$$

as desired. □

With this, we can define the associahedron.

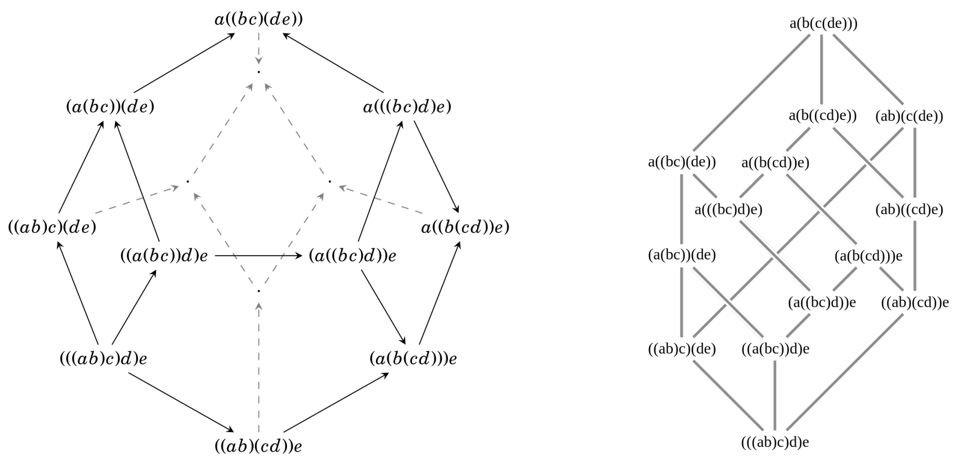
Definition 3.2 (Associahedron)**.** The associahedron $\mathcal{K}^{n-1}$ is the convex hull of the points $M(t)$ in the hyperplane H_n for $t \in Y_n$.

The associahedron $\mathcal{K}^n$ is a polytope of dimension n (see Figure 7a). Since each vertex $M(t)$ of $\mathcal{K}^n$ corresponds to a parenthesization of a word with $n+2$ letters, there are C_{n+1} one of them, where C_k is the kth Catalan number.

One important feature of the associahedron is that it's vertices can be represented as a poset (partially ordered set). We define $s \leq t$ if and only if t can be formed from s by a rightward shift in one set of parentheses.

Example. Take $s = (((x_0 x_1)x_2)x_3)$. Then, take the second parenthesis and the fifth parenthesis to wrap around x_2 and x_3 to get $t = ((x_0 x_1)(x_2 x_3))$. Then, $s \leq t$.

If $s \lessdot t$, that is if $x \leq t$ and there is no z such that $s \leq z \leq t$, then s and t form an edge in the associahedron. Such a poset, whose Hasse diagram is isomorphic to the vertices and edges of $\mathcal{K}^{n-1}$ is known as the *Tamari lattice* T_n. As it turns out, we can similarly construct a partial order on the permutohedron. This partial order is known as the *weak Bruhat order*.



(a) The associahedron $\mathcal{K}^3$ [5]

(b) Tamari lattice T_4 [11]

Figure 7

4 Conclusion

In this paper, we introduced the concept of a polytope and provided a few examples of common examples of polytopes. We continued to discuss the relation between polytopes and convex sets and proved some important theorems that can be used to describe intersections and lattice points in convex sets. Lastly, we discussed two special polytopes and the combinatorial interpretation of their vertices and faces.

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