TOPICS IN ADDITIVE COMBINATORICS

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Abstract. In this article, we investigate some of the useful results of graph theory and how they apply to additive combinatorics.

1. Introduction

Ramsey theory has played an important role in a plethora of mathematical developments throughout the last century, reaching the area of algebra, combinatorics, analysis and geometry. In this article, I will discuss some introductory ramsey theory, and its application onto proving some topics in additive combinatorics, including Schur's Theorem and Van der Waerden's Theorem.

2. Preliminaries

Definition 2.1. A graph $G = (V, E)$ is an ordered pair such that V is a set of elements, as vertices and $E \subseteq \{\{v_1, v_2\} \mid v_1, v_2 \in V, v_1 \neq v_2\}$ is the set of edges.

Definition 2.2. An edge $e \in E$ is **adjacent** to the vertex v if $v \in e$.

Definition 2.3. Two vertices $v_1, v_2 \in V$ is **adjacent** if there exists an edge $e \in E$ such that $e = \{v_1, v_2\}$.

Definition 2.4. A complete graph of n vertices, K_n, is a graph of n vertices and any two vertices are adjacent.

Definition 2.5. K_∞ denotes a complete graph with infinitely many vertices.

Definition 2.6. For a positive integer n, we denote $[n]$ by the set of elements $\{1, 2, \ldots, n\}$.

Definition 2.7. If a, n are positive integers, we denote $[a]^n$ as the set of sequences $(a_1, a_2, \ldots, a_n)$ where $a_i \in [a]$ for all $1 \leq i \leq n$.

3. Introductory Ramsey Theory

Theorem 3.1 (Finite Ramsey's Theorem)**.** *Let m, n be positive integers. There exists a positive integer k with $m, n < k$, such that any graph with at least k vertices and its edges being colored with red and blue, contains either a monochromatic red K_m or a monochromatic blue K_n.*

Definition 3.2. We define the smallest positive integer k satisfying the conclusion of the theorem above as $R(m, n)$, which is called a **Ramsey number**.

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Proof. We will now prove that $R(m, n)$ exists by proving that it is bounded by a constant. We shall establish several facts regarding Ramsey number.

Fact. $R(m, n) = R(n, m)$ for any positive integer m and n.
This is true because we could just swap the colors.

Claim 3.3. *For any integer $a \geq 1$, $R(a, 1) = a$.*

Proof. Note that K_1 contains zero edges, and therefore any coloring of K_1 must contain a blue K_1. □

Claim 3.4. *For any integer $a \geq 1$, $R(2, a) = R(a, 2) = a$.*

Proof. If all of the edges K_a is colored red, it must contain a red K_a. Otherwise, it must contain at least a blue edge, which is a blue K_2. □

Claim 3.5. *If $m, n \geq 2$, then $R(m, n) \leq R(m-1, n) + R(m, n-1)$.*

Proof. We will prove this by induction on $m+n$. Assume that $R(m-1, n)$ and $R(m, n-1)$ exists, and let $X = R(m-1, n) + R(m, n-1)$. We will prove that K_X will contain a monochromatic red K_m or a monochromatic blue K_n. Take a red-blue coloring of K_X and choose a vertex v_0. If v_0 is incident with at least $R(m-1, n)$ red edges, then the complete graph determined by their other endpoints either contains a blue K_n (and we're done) or a red K_{m-1}. Then add vertex v_0 to K_{m-1} to get a red K_m. If not, then v_0 is incident with at least $R(m, n-1)$ blue edges and the idea is identical. This forces $R(m, n) \leq R(m, n-1) + R(m-1, n)$ since we have proved that if $X = R(m, n-1) + R(m-1, n)$, then K_X contains either a monochromatic red K_m or a monochromatic blue K_n. □

□

Although it is straightforward to establish Finite Ramsey's Theorem and Ramsey number bound for $R(m, n)$, for all $m, n \in \mathbb{N}$, it is really difficult to determine its exact value as the bound established is really loose.

In particular, it is known that $R(3, 3) = 6, R(4, 4) = 18, R(5, 5) \in [43, 48]$ and $R(10, 10) \in [798, 23556]$, which is really far from being precise.

There exists an interesting generalization of the Finite Ramsey's Theorem, that is, the Infinite Ramsey's Theorem. Before we state the main theorem, we will first recall the infinite pigeonhole principle (which will just be stated as pigeonhole principle throughout this article).

Theorem 3.6 (Infinite Pigeonhole Principle)**.** *If there are a finite number of pigeonholes containing an infinite number of pigeons at least one of the pigeonholes must contain an infinite number of pigeons.*

Theorem 3.7 (Infinite Ramsey's Theorem)**.** *Suppose we color the edges of K_∞ with finitely many colors, then there is an infinite monochromatic induced subgraph.*

Proof. Fix a vertex v_0. By the Pigeonhole Principle, there is a color c_0 and infinitely many vertices $w \neq v_0$ such that the edge $v_0 w$ has color c_0. Let G_1 be the subgraph spanned by these vertices, so all vertices of G_1 are connected to v_0 by an edge of color c_0. Now pick a vertex v_1 of G_1. By the same argument there is an infinite subgraph G_2 of G_1 and a color c_1 such that all vertices of G_2 are connected to v_1 by an edge of color c_1. By induction we construct an infinite decreasing sequence of subgraphs G_n and a sequence of colors c_n as well as a sequence of vertices $v_n \in G_n$ such that each vertex of G_{n+1} is connected to v_n by an edge of color c_n. Note that this implies that the edge connecting v_i and v_j has color c_i if $i < j$. Applying the Pigeonhole Principle again, we get an infinite increasing sequence $\ell_1 < \ell_2 < \dots$ such that $c_{\ell_1} = c_{\ell_2} = \dots$. By construction, the vertices $v_{\ell_1}, v_{\ell_2}, \dots$ span an infinite monochromatic subgraph. □

4. Schur's Theorem

In the 1910's, Schur attempted to prove Fermat's Last Theorem by reducing the equation $x^n+y^n = z^n$ modulo a prime p. Although the attempt was unsuccessful, Schur has established the following classical result, and one of the earliest results in an area now known as **additive combinatorics**.

Theorem 4.1 (Schur's Theorem)**.** *Let n be a positive integer. There is a positive integer N such that for any coloring of $\{1, 2, \dots, N\}$ using n colors the equation $x + y = z$ has a monochromatic solution.*

Proof. We will color the edges of K_N using n colors, by giving the edge ij, where $i < j$, with the color that $j - i$ has in the coloring of $\{1, 2, \dots, N\}$. By **Finite Ramsey's Theorem**, we know that if N is big enough, then there is a monochromatic triangle, say with vertices $i < j < k$. Therefore, $j - i, k - i$ and $k - j$ has the same color. Furthermore, we have

$$(k - i) = (k - j) + (j - i).$$

□

In 1916, Schur proved the following problem by using Schur's Theorem.

Corollary. Let $n > 1$ be a positive integer. Then for all primes $p > s(n)$, for a function $s : \mathbb{N}_{>1} \to \mathbb{N}$, the congruence

$$x^n + y^n \equiv z^n \pmod{p}$$

has a solution in the integers, such that p does not divide xyz.

Proof. Let p be a large prime and let g be a primitive root modulo p. Color the elements in the set $\{1, 2, \dots, p - 1\}$ with n colors, by giving an element $a \equiv g^j \pmod{p}$ the color $j \pmod{n}$. If p is large, Schur's Theorem yields a, b, c of the same color such that $a = b + c$. Now, since $a \equiv g^i, b \equiv g^j, c \equiv g^k \pmod{p}$. We conclude that

$$g^{i-j} \equiv 1 + g^{k-j} \pmod{p}.$$

However, since a, b, c are of the same color, we know that $i \equiv j \equiv k \pmod{n}$, which implies $n \mid j - i$ and $n \mid k - i$, and therefore, we have

$$(g^{\frac{j-i}{n}})^n \equiv 1^n + (g^{\frac{k-i}{n}})^n \pmod{p}$$

, and we are done. □

5. Van der Waerden's Theorem

In the 1920's, Van der Waerden proved the following result about monochromatic arithmetic progressions in the integers.

Theorem 5.1 (Van der Waerden's Theorem)**.** *For all positive integers k and r, there exists a least integer $W(k, r)$ such that any r-coloring of $[W(k, r)]$ contains a k-term monochromatic arithmetic progression.*

Definition 5.2. We define a set $A \subseteq \mathbb{Z}$ to have **positive upper density** if

$$\lim_{N\to\infty} \sup \frac{|A \cap \{-N, \ldots, N\}|}{2N + 1} > 0$$

In the 1930's, Erdos and Turan conjectured a stronger statement, that any subset of the integers with positive density contains arbitrarily long arithmetic progressions.

In the 1970's, Szemerédi fully settled the conjecture using combinatorial techniques, which has now became one of the landmark theorems in additive combinatorics.

Theorem 5.3 (Szemerédi's theorem)**.** *Every subset of the integers with positive upper density contains arbitrary long arithmetic progressions.*

We will not go through the proof of Szemeredi's theorem in this article. Instead, we will discuss the proof of Van der Waerden's Theorem.

We will use the idea of *color-focusing*. Let $A_1, A_2, \ldots, A_s$ be disjoint arithmetic progressions of length $k - 1$, where $A_i = \{a_i, a_i + d_i, \ldots, a_i + (k - 2)d_i\}$. We define A_i to be **focused** at f if for all i, we have $a_i + (k - 1)d_i = f$. Furthermore, if in some coloring, each A_i is monochromatic, with each A_i receiving a different color, then the progressions together are said to be **color-focused** at f. The key idea is that when $s = r$, an r-coloring of $\mathbb{N}$ containing a set of r color-focused arithmetic progressions $A_1, A_2, \ldots, A_r$, each of length $k - 1$, must contain a monochromatic progressions of length k. Since the common focus f of the A_i must receive one of the r colors, an r-coloring of $\mathbb{N}$ containing a set of r color-focused arithmetic progressions $A_1, A_2, \ldots, A_r$ of length $k-1$, must contain a monochromatic arithmetic progression of length k.

We will proceed by double induction.
By the pigeon hole principle, note that $W(2, r) > r$ because we could take all r elements to be in different color. To prove that $W(2, r) = r + 1$, we could see that by pigeonhole principle, there must exists two of the $r+1$ numbers that have the same color, which is what we wanted – therefore $W(2, r) = r+1$ for all r. Next, suppose that we know that $W(k-1, t)$ is finite for all $t \in \mathbb{N}$. We will show that for a fixed value of r, the value $W(k, r)$ is also finite. To do this, we will show that for each $s \leq r$, there exists a number $V(k, r, s)$ such that any r-coloring of $[V(k, r, s)]$ contains either

- a monochromatic arithmetic progression of length k, or
- A set $A_1, A_2, \ldots, A_s$ of color-focused $(k-1)$ term monochromatic arithmetic progressions, together with their common focus.

For the case $s = 1$, we could just take $V(k, r, 1)$ to be $2 \cdot W(k - 1, r)$. Now, suppose that $V(k, r, s - 1)$ is finite.

Claim 5.4. $V(k, r, s) \leq 2 \cdot V(k, r, s - 1) \cdot W(k - 1, r^{V(k,r,s-1)})$.

Proof. Suppose we are given an r-coloring of $[N]$, where $N = 2 \cdot V(k,r,s-1) \cdot W(k-1, r^{V(k,r,s-1)})$. We break the coloring up into $2 \cdot W = 2W(k-1, r^{V(k,r,s-1)})$ blocks of length $V = V(k,r,s-1)$. Now, there are indeed r^V ways to color each block, so by construction (which follows from the induction hypothesis on k), there is a progression of identically colored blocks $B_i, B_{i+m}, \ldots, B_{i+(k-2)m}$ of length $k-1$ among the first W blocks, whose kth term is also among the $2W$ blocks colored.

Now we look inside each (identically colored) block $B_{\ell+jm}$. By hypothesis (this is the induction on s), we can find $s-1$ color-focused progressions of length $k-1$, together with their focus, within each such block. Suppose that, in color i (where $1 \le i \le s-1$) and in block $\ell + jm$, where $0 \le j \le k-2$, the progression is

$$\{a_i + jmV, a_i + d_i + jmV, \ldots, a_i + (k-2)d_i + jmV\}$$

with focus $f + jmV$.
Unless we have a monochromatic k-term progression, all of these focuses $f + jmV$ for $0 \le j \le k-2$, are colored with a new color w. Now, writing

$$A_i = \begin{cases} \{a_i, a_i + (d_i + mV), a_i + 2(d_i + mV), \ldots, a_i + (k-2)(d_i + mV)\} & 1 \le i \le s-1 \\ \{f, f + mV, f + 2mV, \ldots, f + (k-2)mV\} & i = s, \end{cases}$$

we observe that $A_1, A_2, \ldots, A_s$ form a set of s color-focused progressions of length $k-1$, with common focus $f + (k-1)mV$, which is not greater than N. This completes the first induction on s. For the outer induction on k, note that by the argument at the start of this proof, we must have $W(k,r) \le V(k,r,r)$, and we are hence done. □

Van der Waerden's Theorem cannot be extended to the infinite case, that is, there exists an r-coloring of $\mathbb{N}$, for $r > 1$, under which there is no monochromatic infinite arithmetic progression. We will give a construction for $r = 2$.

Consider the following coloring: where we color 1 red, 2, 3 blue, 4, 5, 6 red, and so on. The coloring produced in this way would look like

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$$

We will label these monochromatic blocks as B_k, where k is the kth block of color. As an example, $B_3 = \{4, 5, 6\}$. Now, we label every number in $\mathbb{N}$ as $f(x,y)$ to label the yth number on the xth block B_x. As an example, $f(1,3) = 4$ because $f(1,3)$ denotes the first number of the third block: in this case is the block $B_3 = \{4, 5, 6\}$. Suppose that there exists a monochromatic infinite arithmetic progression. We take two of its successive elements, let it be $f(a,b)$ and $f(c,d)$. Therefore the difference must be

$$f(a,b) - f(c,d) = \left(\frac{a(a+1)}{2} + b\right) - \left(\frac{c(c+1)}{2} + d\right)$$

which is a constant since $f(a,b)$ and $f(c,d)$ are two successive elements of an arithmetic progression. Since this is a constant, there must exists a natural number ℓ such that

$$\ell > \left(\frac{a(a+1)}{2} + b\right) - \left(\frac{c(c+1)}{2} + d\right) = C$$

Now, consider B_ℓ, where B_ℓ has a different color than the color of infinite monochromatic arithmetic progression

Now, we claim that for any positive integer x, then the interval $[x, x+C]$ must have an element in common with the infinite arithmetic progression. Now, suppose otherwise. Since the arithmetic progression has a common difference of C and the arithmetic progression is of infinite length, we can take the largest element y on the arithmetic progression such that $y < x$. Note that $y \geq x - C$, or otherwise, $(y + C) < (x - C) + C = x$, which is a larger element of the arithmetic progression. Therefore, we have

$$x - C \leq y < x \implies x \leq y + C < x + C$$

As $y + C$ is also the member of the infinite arithmetic progression, and $y + C \in [x, x + C]$, we obtain a contradiction.

To finish this proof, notice that $\ell > C$ by construction and B_ℓ contains ℓ consecutive elements – however, we define B_ℓ to have a different color than the color of our infinite monochromatic arithmetic progression, in which by our previous lemma this is false.

6. References

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