Introduction to the Probabilistic Method

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Abstract

In this article, we get an introduction to the Probabilistic method and how they apply to the field of combinatorics.

1 Introduction

Erdös is known as the pioneer of the probabilistic method, due to his seminal 1947 paper. It is recognized as one of the most important techniques in the field of combinatorics. The probabilistic method is an efficient non constructive method, used to prove the existence of combinatorial objects having some specific properties. It is based on probability theory but, surprisingly, it can be used for proving theorems that have nothing to do with probability. We construct an appropriate probability space and show that a randomly chosen element in this space has desired properties with positive probability.
In this paper, I will introduce the probabilistic method, the Ramsey theory and how the probabilistic method works in its basic form.

2 Ramsey Theory

Theorem 2.1. Ramsey 1930: *For any two integers k,l, there exists R(k,l) such that any two-coloring of the edges of complete graph with N vertices contains either/or*
- *Red complete graph of size k*
- *Green complete graph of size l*

The theorem states that any large graph contains either a clique or an independent set of a given size.

Definition 2.2. Ramsey Number: Let the *Ramsey number* R(k,l) be the smallest n such that if we color the edges of K_n (the complete graph on n vertices) red or blue, we always have a K_k that is all red or a K_l that is all blue.

The Ramsey theorem guarantees that R(k, l) is always finite. However, the precise values of R(k,l) are still unknown but for a few cases, and it is desirable at least to estimate R(k, l) for large k and l. Here we use the *probabilistic method* to prove a lower bound on R(k, k).

Theorem 2.3. For any $k \geq 3$,

$$R(k,k) > 2^{k/2-1}. \tag{1}$$

Proof. Let us consider a random graph G(n,1/2) on n vertices where every pair of vertices forms an edge with probability 1/2, independently of the other edges. (We can imagine flipping a coin for every potential edge to decide whether it should appear in the graph.) For any fixed set of k vertices, the probability that they form a clique is

$$p = 2^{-\binom{k}{2}} \tag{2}$$

The same goes for the occurrence of an independent set, and there are $\binom{n}{k}$ k-tuples of vertices where a clique or an independent set might appear. Now we use the fact that the probability of a union of events is less than or equal to the sum of their respective probabilities, and we get

$$P[G(n,1/2) contains an indep. set of size k] \leq 2\binom{n}{k}2^{-\binom{k}{2}} \tag{3}$$

It remains to choose n so that the last expression is below 1. Using the simplest estimate $\binom{n}{k} \leq n^k$, we find that it is sufficient to have $2n^k < 2^{k(k-1)/2}$. This certainly holds whenever $n \leq 2^{k/2-1}$. Therefore, there are graphs on $\lfloor 2^{k/2-1} \rfloor$ vertices that contain neither a clique of size k nor an independent set of size k. This implies $R(k,k) > 2^{k/2-1}$.

By using finer estimates in the proof, the lower bound for R(k, k) can be improved a little, to $2^k/2$. But a result even slightly better than this seems to require a more powerful technique. In particular, no lower bound is known of the form c^k with a constant $c > \sqrt{2}$, although the best upper bound is about 4^k.

In effect, we are counting the number of bad objects and trying to prove that it is less than the number of all objects, so the set of good objects must be nonempty. In simple cases, it is possible to phrase such proofs in terms of counting bad objects. However, in more sophisticated proofs, the probabilistic formalism becomes much simpler than counting arguments. Furthermore, the probabilistic framework allows us to use many results of probability theory.

For many important problems, the probabilistic method has provided the only known solution, and for others, it has provided accessible proofs in cases where constructive proofs are extremely difficult.

□

3 Hypergraph Coloring

Definition 3.1. A k-uniform hypergraph is a pair (X, S) where X is the set of vertices and $S \subseteq \binom{X}{k}$ is the set of edges (k-tuples of vertices).

Definition 3.2. A hypergraph is c-colorable if its vertices can be colored with c colors so that no edge is monochromatic (at least two different colors appear in every edge).

Definition 3.3. Let m(k) denote the smallest number of edges in a k- uniform hypergraph that is not 2-colorable.

Theorem 3.4. For any $k \geq 2$

$$m(k) \geq 2^{k-1}. \tag{4}$$

Proof. Let us consider a k-uniform hypergraph H with less than 2^{k-1} edges. We will prove that it is 2-colorable.

We color every vertex of H independently red or blue, with probability 1/2. The probability that the vertices of a given edge are all red or all blue is $p = 2 \cdot (\frac{1}{2})^k$. Supposing H has $|S| < 2^{k-1}$ edges, the probability that there exists a monochromatic edge is at most $p|S| < p2^{k-1} = 1$. So there is a non-zero probability that no edge is monochromatic and a proper coloring must exist.

Note that for $k = 3$, we get $m(3) \geq 4$. On the other hand, the smallest known 3-uniform hypergraph that is not 2-colorable is the finite projective plane with 7 points, the *Fano Plane*.

□

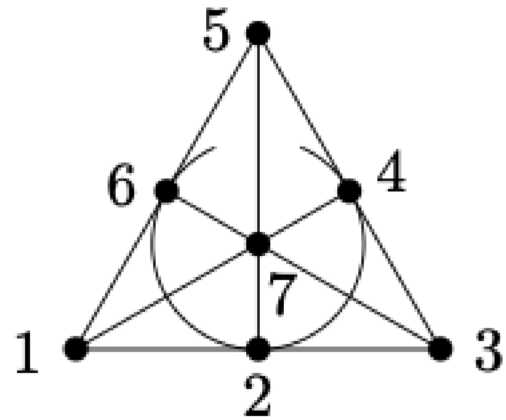
Definition 3.5. The Fano plane is the hypergraph H = (X, S), where

$$X = \{1,2,3,4,5,6,7\}$$

are the points and

$$S = \{\{1,2,3\},\{3,4,5\},\{5,6,1\},\{1,7,4\},\{2,7,5\},\{3,7,6\},\{2,4,6\}\}$$

are the edges.



Theorem 3.6. *Lemma.* $m(3) \leq 7$

Proof. We prove that the Fano plane is not 2-colorable. We use the fact that H is a projective plane, and thus for any two points, there is exactly one edge (line) containing both of them.

Suppose that we have a 2-coloring $A_1 \cup A_2 = X, A_1 \cap A_2 = \phi$, where A_1 is the larger color class.

If $|A_1| \geq 5$, then A_1 contains at least $\binom{5}{2} = 10$ pairs of points. Each pair defines a unique line, but as there are only 7 lines in total, there must be two pairs of points defining the same line. So we have three points of the same color on a line.

If $|A_1| = 4$ then A_1contains $\binom{4}{2} = 6$ pairs of points. If two pairs among them define the same line, that line is monochromatic and we are done. So suppose that these 6 pairs define different lines $l_1, ..., l_6$. Then each point of A_1 is intersected by 3 of the l_i. But since each point in the Fano plane lies on exactly 3 lines and there are 7 lines in total, there is a line not intersecting A_1 at all. That line is contained in A_2 and thus monochromatic.

Now we will improve the lower bound to establish that $m(3) = 7$. □

Theorem 3.7. *Any system of 6 triples is 2-colorable; i.e.* $m(3) \geq 7$

Proof. Let us consider a 3-uniform hypergraph $H = (X, S), |S| \leq 6$. We want to prove that H is 2-colorable. We will distinguish two cases, depending on the size of X.

If $|X| \leq 6$, we apply the probabilistic method. We can assume that $|X| = 6$,because we can always add vertices that are not contained in any edge and therefore do not affect the coloring condition. Then we choose a random subset of 3 vertices which we color red and the remaining vertices become blue. The total number of such coloring's is $\binom{6}{3} = 20$. For any edge (which is a triple of vertices),there are two coloring's that make it either completely red or completely blue,so the probability that it is monochromatic is $\frac{1}{10}$. We have at most 6 edges, and so the probability that any of them is monochromatic is at most $\frac{6}{10} < 1$.

For $|X| > 6$, we proceed by induction. Suppose that $|X| > 6$and $|S| \leq 6$. It follows that there exist two vertices $x, y \in X$ that are not "connected" (a pair of vertices is connected if they appear together in some edge). This is because every edge produces three connected pairs, so the number of connected pairs is at most 18. On the other hand, the total number of vertex pairs is at least $\binom{7}{2} = 21$, so they cannot all be connected.

Now if $x, y \in X$ are not connected, we define a new hypergraph by merging x and y into one vertex:
$X' = X \setminus \{x, y\} \cup \{z\}$,
$S' = \{M \in S : M \cap \{x, y\} = \phi\} \cup \{M \setminus \{x, y\} \cup \{z\} : M \in S, M \cap \{x, y\} \neq \phi\}$.

This (X', S') is a 3-uniform hypergraph as well, $|S'| = |S| \leq 6$, and $|X'| = |X| - 1$, so by the induction hypothesis it is 2-colorable. If we extend the coloring of X' to X so that both x and y get the color of z, we obtain a proper 2-coloring for (X,S)

□

4 The Erdös–Ko–Rado Theorem

Definition 4.1. A family F of sets is intersecting if for all $A, B \in F$, $A \cap B \neq \phi$.

Theorem 4.2. (The Erdös–Ko–Rado Theorem).
If $|X| = n$, $n \geq 2k$, and F is an intersecting family of k-element subsets of X, then

$$|F| \leq \binom{n-1}{k-1} \tag{5}$$

Here the family of all the k-element subsets containing a particular point is intersecting and the number of such subsets is $\binom{n-1}{k-1}$. This configuration is sometimes called a sunflower and the theorem is referred to as the Sunflower Theorem.

Theorem 4.3. ***Lemma**. Consider $X = \{0, 1, ..., n-1\}$ with addition modulo n and define $A_s = \{s, s+1, ..., s+k-1\} \subseteq X$ for $0 \leq s < n$. Then for $n \geq 2k$, any intersecting family $F \subseteq \binom{X}{k}$ contains at most k of the sets A_s*

Proof. If $A_i \in F$,then any other $A_s \in F$ must be one of the sets $A_{i-k+1}, ..., A_{i-1} or A_{i+1}, ..., A_{i+k-1}$. These are 2k - 2 sets, which can be divided into k-1 pairs of the form (A_s, A_{s+k}). As $n \geq 2k$, $A_s \cap A_{s+k} = \phi$,and only one set from each pair can appear in F.

□

Proof. of the theorem. We can assume that $X = \{0, 1, ..., n-1\}$and $F \subseteq \binom{X}{k}$ is an intersecting family. For a permutation $\sigma : X \to X$, we define

$$\sigma(A_s) = \{\sigma(s), \sigma(s+1), ..., \sigma(s+k-1)\} \tag{6}$$

addition again modulo n. The sets $\sigma(A_s)$ are just like those in the lemma, only with the elements relabeled by the permutationσ, so by the lemma at most k of these n sets are in F. Therefore, if we choose random s and σ independently and uniformly,

$$P[\sigma(A_s) \in F] \leq \frac{k}{n} \tag{7}$$

(the underlying probability space here is the product $[n] \times S_n$ with the uniform measure, where S_n is the set of all permutations on [n]). But this choice of $\sigma(A_s)$ is equivalent to a random choice of a k-element subset of X, so

$$P[\sigma(A_s) \in F] = \frac{|F|}{\binom{n}{k}} \tag{8}$$

and

$$|F| = \binom{n}{k} P[\sigma(A_s) \in F] \leq \binom{n}{k} \frac{k}{n} = \binom{n-1}{k-1}. \tag{9}$$

□

References

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