EHRHART POLYNOMIALS

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Abstract. In this paper we study the fundamentals of Ehrhart theory, specifically Ehrhart polynomials and their associated Ehrhart series. Ehrhart polynomials are used to count the number of lattice points in dilates of a convex polytope $\mathcal{P}$. Much of the paper is devoted to building up to a crux of Ehrhart's theory, Ehrhart's theorem for integral polytopes (7).

In order to gain a sense of Ehrhart polynomials, we start by discussing polytopes and the problem of counting lattice points, and then we move on to Pick's theorem. The two next sections discuss simplices, cones, triangulations, and integer-point transforms. All of these sections lay the groundwork for Ehrhart's theorem (7), which enables us to study Ehrhart polynomials in great depth.

The pedagogical approach to this paper was heavily inspired by [BR07]; all proofs are from that text. [Bra07], [Ked20] and [Liu18] were used as general supplementary texts, while [Koh13] was used to flesh out the section on integer-point transforms.

Contents

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1. Introduction

From a combinatorial perspective, we are interested in enumerating the lattice points contained in a given convex polytope $\mathcal{P}$. This count is called the *lattice point enumerator* $L_{\mathcal{P}}$. Ehrhart polynomials establish a relationship between $t\mathcal{P}$ and $L_{\mathcal{P}}(t)$. Theorem 7 says that for a convex integral polytope $\mathcal{P}$, the lattice point enumerator of its tth dilate may be expressed as

$$L_{\mathcal{P}}(t) = c_d t^d + c_{d-1}t^{d-1} + \ldots + c_0.$$

Indeed, this is the *Ehrhart polynomial* of $\mathcal{P}$. Moreover, we may construct an associated generating function, called the *Ehrhart series of* $\mathcal{P}$, that encodes information about $\mathcal{P}$'s lattice points. We define

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t\geq 1} L_{\mathcal{P}}(t)z^t$$

as this generating function. The coefficients of an Ehrhart polynomial also encode some interesting information. Of special interest is the leading coefficient c_d, which is the continuous volume of $\mathcal{P}$. Here, as well as in Pick's theorem (Theorem 4.1), we see a nice connection between the continuous Riemannian volume and the point-defined discrete volume of $\mathcal{P}$.

The structure of the paper is as follows. The Sections 2, 3, and 4 are introductory. We start off by establishing the fundamentals of polytopes in Section 2, then move on to the problem of lattice point enumeration in Section 3 and finally an elementary 2-dimensional case of Ehrhart polynomials, Pick's theorem (Theorem 4.1), in Section 4. In Sections 5 and 6, we discuss simplices, cones, and triangulations and integer-point transforms, respectively, in preparation for the proof of Ehrhart's theorem for integral polytopes (7). Finally, we interpret the coefficients of Ehrhart polynomials and provide a proof of Pick's theorem in Section 8.

2. Polytopes

In order to get well-versed in the language of Ehrhart polynomials, we must first understand the language of polytopes. Intuitively, we may think of a polytope $\mathcal{P}$ as a generalization of polygons and polyhedra to an arbitrary number of dimensions. We may formally define a polytope, specifically a *convex polytope,* in terms of its *vertex description* and *hyperplane description*, which we'll refer to as its $\mathcal{V}$-description and $\mathcal{H}$-description, respectively.

Definition 2.1 ($\mathcal{V}$-description)**.** A *convex polytope* is the convex hull of finitely many points in $\mathbb{R}^d$. That is, for any finite set of points $V = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}\} \subset \mathbb{R}^d$,

$$\mathcal{P} := \mathrm{conv}(V) = \{\lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \ldots + \lambda_n\mathbf{v_n} : \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_n = 1\}$$

Definition 2.2. The *dimension* of a polytope $\mathcal{P}$ is the dimension of the affine space

$$\mathrm{span}\ \mathcal{P} := \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in \mathcal{P}, \lambda \in \mathbb{R}\}$$

spanned by $\mathcal{P}$. If $\mathcal{P}$ has dimension d, we say that $\mathcal{P}$ is a d-polytope. Note that $\mathcal{P} \subset \mathbb{R}^d$ does not necessarily have dimension d.

By the definition, a convex polytope is a closed subset of $\mathbb{R}^d$. Alternatively, we may think of a convex polytope as being defined by bounded intersections of finitely many half-spaces and hyper-planes.

Definition 2.3. A *half-space* in $\mathbb{R}^n$ is the set of solutions to a linear inequality of the form $\mathbf{a} \cdot \mathbf{x} \leq b$ for some $a \in \mathbb{R}^n, b \in R$.

Definition 2.4. A *hyperplane* in $\mathbb{R}^n$ is the set of solutions to a linear equation of the form $\mathbf{a} \cdot \mathbf{x} = b$ for some $a \in \mathbb{R}^n, b \in R$.

A polytope's ability to be expressed as its $\mathcal{V}$-description or $\mathcal{H}$-description gives us two different nice ways of viewing it. The proof of this fact is non-trivial, but it is quite lengthy, so we invite you to look at Appendix A of [BR07] for it.

Example 2.1. Consider a simple example of a 2-dimensional polytope $\mathcal{P}$, in this case a triangle.

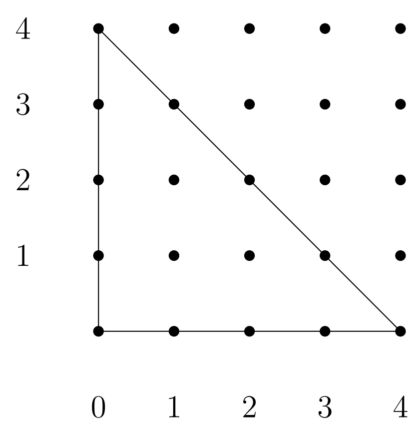

Figure 1. A triangular polytope.

By inspection, we see that $\mathcal{P}$'s $\mathcal{H}$-description is $\mathcal{P} = \{x, y : x \geq 0, y \geq 0, x + y \leq 4\}$, and its $\mathcal{V}$-description is $\text{conv}(\{(0,0),(4,0),(0,4)\}$. $\square$

Let us introduce some other important properties of polytopes that originate from the concept of a supporting hyperplane.

Definition 2.5. Given a convex polytope $\mathcal{P} \subset \mathbb{R}^d$, the hyperplane $\mathcal{H} = \{x \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$ is a *supporting hyperplane* of $\mathcal{P}$ if $\mathcal{P}$ lies entirely on one side of $\mathcal{H}$. That is, $\mathcal{P} \subset \{x \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \leq b\}$ or $\mathcal{P} \subset \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \geq b\}$.

A *face* of $\mathcal{P}$ is a set of the form $\mathcal{P} \cap \mathcal{H}$, where $\mathcal{H}$ is a supporting hyperplane of $\mathcal{P}$. Note that $\mathcal{P}$ itself is a face of $\mathcal{P}$, which corresponds to the degenerate hyperplane $\mathbb{R}^d$, as is the empty set $\emptyset$, which corresponds to a hyperplane that does not meet $\mathcal{P}$. *(d-1)*-dimensional faces are called *facets*, the 1-dimensional faces *edges*, and the 0-dimensional faces *vertices* of $\mathcal{P}$. For example, a 3-dimensional cube has 8 vertices, 12 edges, and 6 facets.
We also introduce some special kinds of polytopes that are of interest.

Definition 2.6. A convex polytope $\mathcal{P}$ is *integral* if all of its vertices have integer coordinates.

Definition 2.7. A convex polytope $\mathcal{P}$ is *rational* if all of its vertices have rational coordinates.

Definition 2.8. A convex d-polytope with exactly $d+1$ vertices is called a *d-simplex*.

Note that all polytope have at least $d+1$ vertices. Every 1-dimensional polytope is a 1-simplex, namely, a line segment. A 2-dimensional simplex is a triangle, a 3-dimensional simplex is a tetrahedron, and a 4-dimensional simplex is a 5-cell. We shall encounter simplices in greater depth when we deal with triangulations in Section 5.

3. Counting Lattice Points

Often we are interested in computing how many lattice points are contained inside of a polytope. Let's establish a few definitions first.

Definition 3.1. The d-dimensional *integer lattice* $\mathbb{Z}^d \subset \mathbb{R}^d$ is the collection of all points with integer coordinates in $\mathbb{R}^d$.

Unless otherwise specified, assume that all lattices in this paper are integer lattices.

Definition 3.2. *Lattice points* are points in $\mathbb{Z}^d$; that is, they are points in $\mathbb{R}^d$ such that all of their coordinates are integers.

Although this paper is approachable with informal ideas of what "interior points" and "boundary points" are, we will formalize the terms here according to their topological definitions.

Definition 3.3. The *interior* of a polytope $\mathcal{P}$ is the union of all its open subsets. An *interior point* lies within the space formed by this union.

Definition 3.4. A *boundary point* is a point that is a member of the set closure of a given set S and the set closure of its complement set. If S is a subset of $\mathbb{R}^n$, then a point x in $\mathbb{R}^n$ is a boundary point of S if every neighborhood of x contains at least one point in S and at least one point not in S.

Definition 3.5. The polygon $t\mathcal{P}$ is the polytope resulting from scaling $\mathcal{P}$ by a factor of t. We call this new polytope the tth dilate of $\mathcal{P}$.

Definition 3.6. For the t^{th} dilates of a convex polytope $\mathcal{P} \subset \mathbb{R}^d$, we denote the *lattice point enumerator* $L_{\mathcal{P}}(t)$, also called the *discrete volume of $t\mathcal{P}$,* by

$$L_{\mathcal{P}}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^d)$$

We may also think of fixing $\mathcal{P}$ and shrinking the integer lattice, so that we have

$$L_{\mathcal{P}}(t) = \#(\mathcal{P} \cap \frac{1}{t}\mathbb{Z})^d.$$

Note that $L_{\mathcal{P}}(t)$ includes both the interior points and boundary points of $t\mathcal{P}$.

Example 3.1. A classic introductory example of a polytope is the unit d-cube $\mathcal{Q} = [0,1]^d$. The vertex description of $\mathcal{Q}$ is given by the set of 2^d vertices

$$\{(x_1, x_2, \ldots, x_d) : x_k = 0 \text{ or } 1\}$$

and the hyperplane description is

$$\mathcal{Q} = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_k \leq 1 \text{ for all } k = 1, 2, \ldots, d\}$$

So we have $2d$ bounding hyperplanes defined by $x_1 = 0, x_1 = 1, x_2 = 0, x_2 = 1, \ldots, x_d = 0, x_d = 1$.

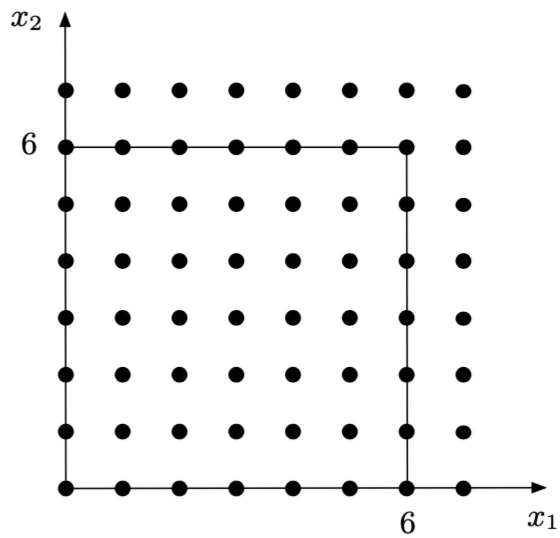

Figure 2. The sixth dilate of $\mathbb{Q}$ in $\mathbb{R}^2$ (from [BR07]).

What is the discrete volume of $\mathcal{Q}$? It is easy to just dilate by $t \in \mathbb{Z}^+$ and count:

$$\#(t\mathcal{Q} \cap \mathbb{Z}^d) = ([0,t]^d \cap \mathbb{Z}^d) = (t+1)^d.$$

Recalling our previous notation for discrete volume, notice that $L_{\mathcal{Q}}(t) = (t+1)^d = \sum_{k=0}^{d} \binom{d}{k} t^k$, a polynomial in the integer variable t. Its coefficients are just the binomial coefficients $\binom{d}{k}$. This is not a unique occurrence, as we shall see later on.

What about the interior $\mathcal{Q}^\circ$ of the cube? Interestingly enough, the number of interior lattice points in $t\mathcal{Q}^\circ$ is

$$L_{\mathcal{Q}^\circ}(t) = \#(t\mathcal{Q}^\circ \cap \mathbb{Z}^d) = ((0,t)^d \cap \mathbb{Z}^d) = (t-1)^d$$

□

Observe that this polynomial equals $(-1)L_{\mathcal{Q}}(-t)$, the evaluation of the polynomial $L_{\mathcal{Q}}(t)$ at negative integers, up to a sign. We will discuss this fact, called the *Ehrhart-Macdonald reciprocity*, in more detail when we dive into Ehrhart polynomials in Section 7.

To close off this section, we will introduce the concept of an Ehrhart series. An important tool for analyzing any polytope $\mathcal{P}$, it is the generating function of $L_{\mathcal{P}}(t)$.

Definition 3.7. The Ehrhart series of a convex polytope $\mathcal{P}$ is

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t.$$

Ehrhart series appear often in this paper, and serve as a method for encoding information about $L_{\mathcal{P}}(t)$ in a generating function.

4. Pick's Theorem

We now return to $\mathbb{R}^2$ for some lattice point enumeration. Consider an integer lattice. Denote the number of lattice points inside the polygon $\mathcal{P}$ by I, the number of lattice points on the boundary of $\mathcal{P}$ by B, and the polygon's area by A. The following result due to Pick accounts for all $L_{\mathcal{P}}$ in $\mathbb{R}^2$.

Theorem 4.1 (Pick)**.** For a integral convex polygon $\mathcal{P}$,

$$A = I + \frac{B}{2} - 1$$

There are several proofs of Pick's theorem, many of which involve detailed casework. We invite you to look at [Gar10] for a nice proof involving elementary graph theory and the Euler characteristic formula. At the end of the paper, we will present a much more elegant proof of Pick's theorem using Ehrhart polynomials.

Example 4.2. Consider the polygon below, which we'll call $\mathcal{B}$ for its perverted bow tie shape.

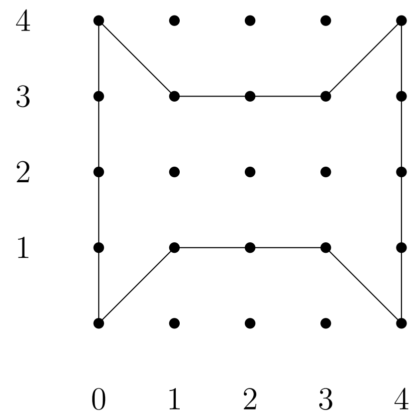

Figure 3. A bow tie-shaped polytope in $\mathbb{R}^2$.

In this case, $I = 3$ and $B = 16$. Therefore $A_{\mathcal{B}} = 3 + \frac{16}{2} - 1 = 10$ square units. □

A logical next question is whether Pick's theorem extends to $\mathbb{R}^d$ for $d \geq 3$. Using his eponymous tetrahedron, Reeve (see [Ree57]) proved that the answer to this question is no.

5. Simplices and Cones

Before we fully plunge into Ehrhart polynomials, we must first explore the concepts of simplices and cones, as they will help us prove an important result, namely Theorem 7.1. Recall that a simplex Δ is a convex d-polytope with exactly $d+1$ vertices. We may also define the notion of the standard simplex. Though this term is not particularly important for our purposes, it is good to know.

Definition 5.1. The d-dimensional *standard simplex* Δ is the convex hull of the $d+1$ points $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ and the origin. Here, $\mathbf{e}_j$ is the unit vector $(0, \ldots, 1, \ldots, 0)$, with a 1 in the jth position.

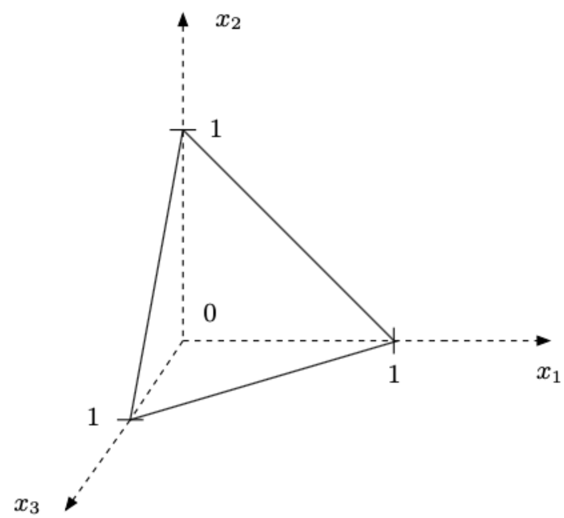

Figure 4. The standard simplex Δ in $\mathbb{R}^3$ (from [BR07]).

The process of decomposing a polytope into simplices proves to be very useful in the proof of Theorem 7.1; this is called *triangulating*. We define a triangulation as follows.

Definition 5.2. A *triangulation* of a convex d-polytope $\mathcal{P}$ is a finite collection T of d-simplices with the following properties:

- $\mathcal{P} = \bigcup_{\Delta \in T} \Delta$.
- For any $\Delta_1, \Delta_2 \in T, \Delta_1 \cap \Delta_2$ is a face common to both Δ_1 and Δ_2.

We say that $\mathcal{P}$ can be *triangulated using no new vertices* if there exists a triangulation T such that the vertices of any $\Delta \in T$ are vertices of $\mathcal{P}$.

Theorem 5.1 (Existence of triangulations)**.** Every convex polytope can be triangulated using no new vertices.

Although this theorem seems intuitively obvious, it is not trivial to prove. However, the proof is quite involved, so we invite you to look at Appendix B of [BR07].

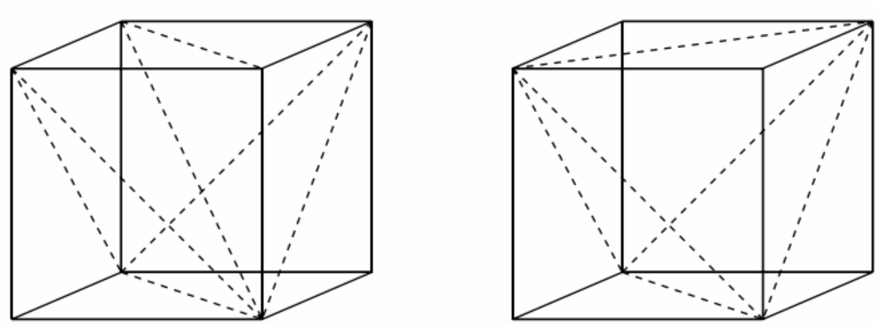

Figure 5. Two different triangulations of a 3-cube (from [BR07])

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Definition 5.3. A *pointed cone* $\mathcal{K} \subseteq \mathbb{R}^d$ is a set of the form

$$\mathcal{K} = \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \ldots + \lambda_m \mathbf{w}_m : \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\},$$

where $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{R}^d$ are such that there exists a hyperplane $\mathcal{H}$ for which $\mathcal{H} \cap \mathcal{K} = \{v\}$; that is, $\mathcal{K} \backslash \{v\}$ lies strictly on one side of $\mathcal{H}$.

In this case the vector $\mathbf{v}$ is called the *apex* of $\mathcal{K}$, and the $\mathbf{w}_k$'s are the *generators* of $\mathcal{K}$. The cone is *rational* if $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{Q}^d$, in which case we may choose $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{Z}^d$ by clearing denominators. The *dimension* of $\mathcal{K}$ is the dimension of the affine space spanned by $\mathcal{K}$. If $\mathcal{K}$ is of dimension d, we call the cone a d-cone. We call the d-cone $\mathcal{K}$ *simplicial* if $\mathcal{K}$ has precisely d linearly independent generators.

Just as polytopes can be expressed as an intersection of half-spaces, so can pointed cones. A rational pointed d-cone is the d-dimensional intersection of finitely many half-spaces of the form

$$\{\mathbf{x} \in \mathbb{R}^d : a_1 x_1 + a_2 x_2 + \ldots + a_d x_d \leq b\}$$

with integral parameters $a_1, a_2, \ldots, a_d, b \in \mathbb{Z}$ such that the corresponding hyperplanes of the form

$$\{\mathbf{x} \in \mathbb{R}^d : a_1 x_1 + a_2 x_2 + \ldots + a_d x_d = b\}$$

meet in exactly one point.

One of the most practical uses of cones for our purposes is called *coning a polytope*. Given a convex polytope $\mathcal{P} \subset \mathbb{R}^d$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we lift those vertices into $\mathbb{R}^{d+1}$ by adding a 1 as their last coordinate. Therefore, let

$$\mathbf{w}_1 = (\mathbf{v}_1, 1), \mathbf{w}_2 = (\mathbf{v}_2, 1), \ldots, \mathbf{w}_n = (\mathbf{v}_n, 1).$$

Using these new vertices, we may define the *cone over* $\mathcal{P}$.

Definition 5.4. The *cone over* $\mathcal{P}$ is defined as

$$\text{cone}(\mathcal{P}) = \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \ldots + \lambda_n \mathbf{w}_n : \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\} \subset \mathbb{R}^d.$$

The pointed cone has the origin as its apex, and we can recover our original polytope $\mathcal{P}$ (strictly speaking, the translated set $\{(\mathbf{x}, 1) : \mathbf{x} \in \mathcal{P}\}$) by cutting the cone($\mathcal{P}$) with the hyperplane $x_{d+1} = 1$, as shown in the figure below.

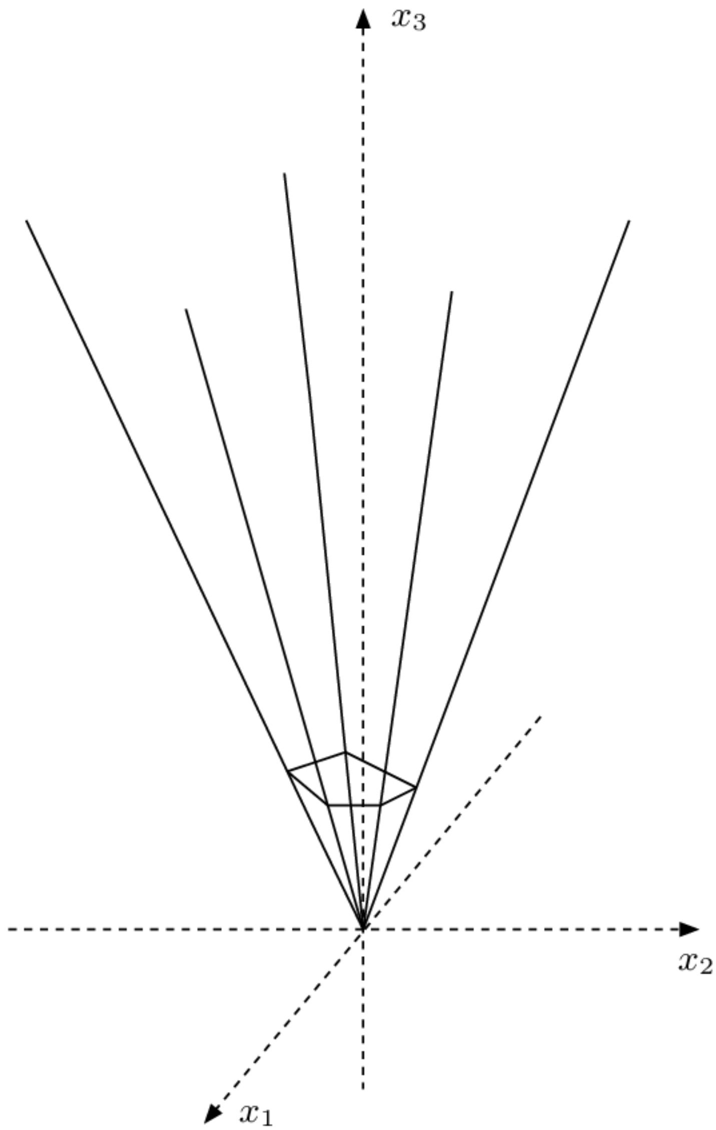

Figure 6. Coning over a polytope (from [BR07]).

There are many similarities between polytopes and cones, several of which we note below.

Definition 5.5. The hyperplane $\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b$ is a *supporting hyperplane* of the pointed d-cone $\mathcal{K}$ if $\mathcal{K}$ lies entirely on one side of $\mathcal{H}$. So $\mathcal{K} \subset \{x \in \mathbb{R}^d : a \cdot x \leq b\}$ or $\mathcal{K} \subset \{x \in \mathbb{R}^d : a \cdot x \geq b\}$.

Our definitions of faces, edges, and facets from Section 2 hold for $\mathcal{K}$. Note that the apex of $\mathcal{K}$ is its unique vertex.

Just as polytopes can be triangulated into simplices, pointed cones can be triangulated into simplicial cones.

Definition 5.6. A collection T of simplicial d-cones is a *triangulation* of the d-cone $\mathcal{K}$ if it satisfies:

- $\mathcal{K} = \bigcup_{\mathcal{S} \in T} S$.

- For any $S_1, S_2 \in T, S_1 \cap S_2$ is a face common to both S_1 and S_2.

We say that $\mathcal{K}$ can be *triangulated using no new generators* if there exists a triangulation T such that the generators of any $\mathcal{S} \in T$ are generators of $\mathcal{P}$.

Theorem 5.2. Any pointed cone can be triangulated into simplicial cones using no new generators.

Proof. Given a pointed d-cone $\mathcal{K}$, there exists a hyperplane $\mathcal{H}$ that intersects $\mathcal{K}$ only at the apex. Now translate $\mathcal{H}$ "into" the cone, so that $\mathcal{H} \cap \mathcal{K}$ consists of more than one point. This intersection is a $(d-1)$-polytope $\mathcal{P}$, whose vertices are determined by the generators of $\mathcal{K}$. Now triangulate $\mathcal{P}$ using no new vertices. The cone over each simplex of the triangulation is a simplicial cone. These simplicial cones, by construction, triangulate $\mathcal{K}$. □

6. Integer-Point Transforms for Rational Cones

Here we develop a tool, the integer-point transform, that allows us to encode the integer points of a set as a generating function. Consider a set $S \subset \mathbb{R}^d$, where S may be a rational cone or polytope.

Definition 6.1. The *integer-point transform* of S is σ_S, where

$$\sigma_S(\mathbf{z}) = \sigma_S(z_1, z_2, \ldots, z_d) = \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}.$$

The generating function σ_S simply lists all integer points in S in a special form: not as a list of vectors, but rather as a formal sum of monomials. Note that $\sigma_S =$ also called the *moment generating function* or the *generating function of S*. The integer-point transform σ_S allows for the application of both algebraic and analytical techniques to our study of Ehrhart polynomials.

Definition 6.2. Given a simplicial d-cone $\mathcal{K}$, where

$$\mathcal{K} = \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \ldots + \lambda_n \mathbf{w}_d : \lambda_1, \lambda_2, \ldots, \lambda_d \geq 0\}$$

the *fundamental parallelogram* of K is

$$\Pi = \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \ldots + \lambda_n \mathbf{w}_d : 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_d < 1\}.$$

The fundamental parallelogram is an important notion because taking linear combinations of $\mathcal{K}$'s generators leads to a tiling of the entire cone. It is crucial for the fundamental parallelogram to be half open, since otherwise the tiling would not be disjoint.

Example 6.1. Consider the 1-dimensional cone $\mathcal{K} = [0, \infty)$. Its integer-point transform is

$$\sigma_{\mathcal{K}}(z) = \sum_{m \in [0,\infty) \cap \mathbb{Z}} z^m = \sum_{m \geq 0} z^m = \frac{1}{1-z}.$$

□

Example 6.2. Consider the following cone $\mathcal{K} \subset \mathbb{R}^2$:

$$\mathcal{K} := \{\lambda_1(2,1) + \lambda_2(1,2) : \lambda_1, \lambda_2 \geq 0\}.$$

Observe that the vectors (2,1) and (1,2) are the generators of $\mathcal{K}$. We want to tile the cone by its fundamental parallelogram Π, which has the form

$$\Pi = \{\lambda_1(2,1) + \lambda_2(1,2) : 0 \leq \lambda_1, \lambda_2 < 1\}.$$

These linear combinations can be written as

$$\sum_{j\geq 0, k\geq 0} (x,y)^{j(2,1)+k(1,2)} = \frac{1}{(1-x^2y^1)(1-x^1y^2)}.$$

The geometric interpretation of this statement is that our cone is now tiled by the fundamental parallelogram above. If we take an integer point (m,n) out of Π and add a linear combination of the generators we get a subset of $\mathbb{Z}^2$ that has the form

$$\{(m,n) + j(2,1) + k(1,2) : j,k \in \mathbb{Z}_{\geq 0}\}.$$

The cone $\mathcal{K}$ can be described as the disjoint union of all such subsets, if (m,n) ranges over all lattice points in Π. Since Π contains the lattice points (0,0) and (1,1), the integer-point transform takes on the form

$$\begin{aligned}\sigma_{\mathcal{K}(x,y)} &= (1+xy)\sum_{j\geq 0, k\geq 0} (x,y)^{j(2,1)+k(1,2)} \\ &= \frac{1+xy}{(1-x^2y^1)(1-x^1y^2)}\end{aligned}$$

□

The following result relates the integer-point transform $\sigma_{\mathbf{v}+\mathcal{K}}$ of the shifted cone $\mathbf{v}+\mathcal{K}$ with the integer-point transform $\sigma_{\mathbf{v}+\Pi}$ of the shifted parallelepiped $\mathbf{v}+\Pi$.

Theorem 6.3. Suppose

$$\mathcal{K} = \{\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \ldots + \lambda_n\mathbf{w}_d : \lambda_1, \lambda_2, \ldots, \lambda_d \geq 0\}$$

is a simplicial d-cone, where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$. Then for $\mathbf{v} \in \mathbb{R}^d$, the integer-point transform $\sigma_{v+\mathcal{K}}$ of the shifted cone $\mathbf{v}+\mathcal{K}$ is the rational function

$$\sigma_{v+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{v+\Pi}(z)}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_d})},$$

where Π is the half-open parallelepiped

$$\Pi = \{\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \ldots + \lambda_n\mathbf{w}_d : 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_d < 1\}.$$

□

Proof. In $\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \sum_{\mathbf{m}\in(\mathbf{v}+\mathcal{K})\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$, we list each integer point $\mathbf{m}$ in $\mathbf{v}+\mathcal{K}$ as the monomial $\mathbf{z}^{\mathbf{m}}$. Such a lattice point can, by definition, be written as

$$\mathbf{m} = \mathbf{v} + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \ldots + \lambda_d\mathbf{w}_d$$

for some numbers $\lambda_1, \lambda_2, \ldots, \lambda_d \geq 0$. As the $\mathbf{w}_k$'s form a basis of $\mathbb{R}^d$, this representation is unique. Let's write each of the $\mathbf{w}_k$'s in terms of their integer and fractional parts: $\lambda_k = \lfloor\lambda_k\rfloor + \{\lambda_k\}$. Therefore

$$\mathbf{m} = \mathbf{v} + (\{\lambda_1\}\mathbf{w}_1 + \{\lambda_2\}\mathbf{w}_2 + \ldots + \{\lambda_d\}\mathbf{w}_d) + \lfloor\lambda_1\rfloor\mathbf{w}_1 + \lfloor\lambda_2\rfloor\mathbf{w}_2 + \ldots + \lfloor\lambda_d\rfloor\mathbf{w}_d,$$

and we should note that, since $0 \leq \{\lambda_k\} < 1$, the vector

$$\mathbf{p} := \mathbf{v} + \{\lambda_1\}\mathbf{w}_1 + \{\lambda_2\}\mathbf{w}_2 + \ldots + \{\lambda_d\}\mathbf{w}_d$$

is in $\mathbf{v}+\Pi$. In fact, since $\mathbf{m}$ and $\lfloor\lambda_k\rfloor$ are all integer vectors, $\mathbf{p} \in \mathbb{Z}^d$. Again the representation of $\mathbf{p}$ in terms of the $\mathbf{w}_k$'s is unique. In summary, we have proved that any $\mathbf{m} \in \mathbf{v}+\mathcal{K}\cap\mathbb{Z}^d$ can be uniquely written as

$$\mathbf{m} = \mathbf{p} + k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_d\mathbf{w}_d \tag{6.1}$$

for some $\mathbf{p} \in (\mathbf{v} + \Pi) \cap \mathbb{Z}^d$ and some integers $k_1, k_2, \ldots, k_d \geq 0$. On the other hand, let us write the rational function on the RHS of the theorem as a product of series. Then we have

$$\frac{\sigma_{v+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_d})} = \left(\sum_{\mathbf{p}\in(\mathbf{v}+\Pi)\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}\right)\left(\sum_{k_1\geq 0} \mathbf{z}^{k_1\mathbf{w}_1}\right)\cdots\left(\sum_{k_d\geq 0} \mathbf{z}^{k_d\mathbf{w}_d}\right).$$

If we multiply everything out, a typical exponent will look exactly like equation 6.1. □

There is an important geometric idea behind this proof: we tile the cone $\mathbf{v} + \mathcal{K}$ with translated of $\mathbf{v} + \Pi$ by nonnegative integral combinations of the $\mathbf{w}_k$'s. Because of this tiling, we get the nice integer-point transform in Theorem 6.3. We see that the complexity of computing the integer-point transform $\sigma_{\mathbf{v}+\mathcal{K}}$ is embedded in the location of the lattice points in the parallelepiped $\mathbf{v} + \Pi$.

Having said all this, we are now equipped with some important tools from the past two sections: simplices, cones, triangulations, and integer-point transforms. All of these concepts will aid us in the crown jewel of this paper, the proof of Theorem 7.1.

7. Ehrhart's Theorem

And now onto the main event! We encountered Ehrhart polynomials in Section 2 when we defined discrete volume, we saw a special case of them in Section 4 when we discussed Pick's theorem, and we've spent many sections building up the necessary knowledge base for the proof of Theorem 7.1. Now we can study Ehrhart's theorem, which gives Ehrhart polynomials life.

Theorem 7.1 (Ehrhart's theorem for integral polytopes)**.** *Let $\mathcal{P}$ be a convex integral d-polytope. Then $L_{\mathcal{P}}$ is a polynomial in t of degree d.*

We call the lattice point enumerator $L_{\mathcal{P}}(t)$ the *Ehrhart polynomial* of $\mathcal{P}$. This theorem establishes a powerful relationship between different dilations of $\mathcal{P}$ and allows us to calculate their discrete volumes in a systematic way. We will soon demonstrate a proof of Ehrhart's theorem that uses generating functions. However, we must build a little more machinery first.

Our proof of Theorem 7.1 uses generating functions of the form $\sum_{t\geq 0} f(t)z^t$.

Lemma 7.2. *If*

$$\sum_{t\geq 0} f(t)z^t = \frac{g(z)}{(1-z)^{d+1}},$$

then f is a polynomial of degree d iff g is a polynomial of degree at most d and $g(1) \neq 0$.

Unfortunately, proof of this lemma is beyond the scope of this paper.

We introduced generating functions of the form $\sigma_S(\mathbf{z}) = \sigma_S(z_1, z_2, \ldots, z_d) = \sum_{\mathbf{m}\in S\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$ in the last section because they're extremely useful for lattice-point problems. If we're interested in simply counting lattice points, we simply evaluate σ_S at $\mathbf{z} = (1, 1, \ldots, 1)$:

$$\sigma_S(1, 1, \ldots, 1) = \sum_{\mathbf{m}\in S\cap\mathbb{Z}^d} \mathbf{1}^{\mathbf{m}} = \sum_{\mathbf{m}\in S\cap\mathbb{Z}^d} 1^{\mathbf{m}} = \#(S \cap \mathbb{Z}^d).$$

where $\mathbf{1}$ denotes the vector whose components are all 1. Note that we should only make this evaluation if S is bounded. Now let's introduce a slight modification on this equation that will help us connect integer-point transforms and Ehrhart series.

Lemma 7.3.

$$\sigma_{\mathrm{cone}(\mathcal{P})}(1,1,\ldots,1,z) = 1 + \sum_{t\geq 1} L_{\mathcal{P}}(t)z^t = \mathrm{Ehr}_{\mathcal{P}}(z).$$

Proof. Recall that in order to cone over a convex polytope $\mathcal{P} \subset \mathbb{R}^d$, we lift its vertices into $\mathbb{R}^{d+1}$. If $\mathcal{P}$ has the vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{Z}^d$, we add 1 as their last coordinate. Therefore let

$$\mathbf{w}_1 = (\mathbf{v}_1, 1), \mathbf{w}_2 = (\mathbf{v}_2, 1), \ldots, \mathbf{w}_n = (\mathbf{v}_n, 1).$$

Then

$$\mathrm{cone}(\mathcal{P}) = \{\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \ldots + \lambda_n\mathbf{w}_n : \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\} \subset \mathbb{R}^d.$$

Now let's form the integer-point transform $\sigma_{\mathrm{cone}(\mathcal{P})}$ of cone($\mathcal{P}$). Let's look at different powers of z_{d+1}. There is one term, namely 1, with z_{d+1}^0, that corresponds to the origin. The term ${z_{d+1}}^1$ corresponds to lattice points in $\mathcal{P}$ (listed as monomials in $z_1, z_2, \ldots, z_d$), the terms with ${z_{d+1}}^2$ correspond to points in $2\mathcal{P}$, etc. So we have

$$\begin{aligned}\sigma_{\mathrm{cone}(\mathcal{P})}(z_1, z_2, \ldots, z_{d+1}) &= 1 + \sigma_{\mathcal{P}}(z_1, \ldots, z_d)z_{d+1} + \sigma_{2\mathcal{P}}(z_1, \ldots, z_d){z_{d+1}}^2 + \ldots \\ &= 1 + \sum_{t\geq 1} \sigma_{t\mathcal{P}}(z_1, \ldots, z_d)z_{d+1}^t.\end{aligned}$$

Specializing further for enumeration purposes, we recall that $\sigma_{\mathcal{P}}(1,1,\ldots,1) = \#(\mathcal{P} \cap \mathbb{Z}^d)$, and so

$$\begin{aligned}\sigma_{\mathrm{cone}(\mathcal{P})}(1,1,\ldots,1,z_{d+1}) &= 1 + \sum_{t\geq 1} \sigma_{t\mathcal{P}}(1,1,\ldots,1)z_{d+1}^t \\ &= 1 + \sum_{t\geq 1} \#(t\mathcal{P} \cap \mathbb{Z}^d)z_{d+1}^t\end{aligned}$$

By definition, the enumerators on the RHS are just evaluations of Ehrhart's counting function. That is, $\sigma_{\mathrm{cone}(\mathcal{P})}(1,1,\ldots,1,z_{d+1}$ is nothing but the Ehrhart series of $\mathcal{P}$, so our proof is complete.

□

Now we have all of the machinery needed to prove the centerpiece of this paper, Theorem 7.1.

Proof. It suffices to prove Theorem 7.1 for simplices, because we can triangulate any integral polytope into integral simplices using no new vertices. Note that these simplices will intersect in lower-dimensional integral simplices.

By Lemma 7.2, it suffices to show that for an integral d-simplex Δ,

$$\mathrm{Ehr}_{\Delta}(z) = 1 + \sum_{t\geq 1} L_{\Delta}(t)z^t = \frac{g(z)}{(1-z)^{d+1}}$$

for some polynomial g of degree at most d with $g(1) \neq 0$. In Lemma 7.3, we showed that the Ehrhart series of Δ equals $\sigma_{\mathrm{cone}(\Delta)}(1,1,\ldots,1,z)$, so let's study the integer-point transform attached to cone(Δ).

The simplex Δ has $d+1$ vertices $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{d+1}}$, and so $\mathrm{cone}(\Delta) \subset \mathbb{R}^{d+1}$ is simplicial, with apex the origin and generators

$$\mathbf{w_1} = (\mathbf{v}_1, 1), \mathbf{w_2} = (\mathbf{v}_2, 1), \ldots, \mathbf{w_{d+1}} = (\mathbf{v}_{d+1}, 1) \in \mathbb{Z}^{d+1}.$$

Now we may utilize Theorem 6.3:

$$\sigma_{\text{cone}(\Delta)}(z_1, z_2, \ldots, z_{d+1}) = \frac{\sigma_\Pi(z_1, z_2, \ldots, z_{d+1})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_{d+1}})},$$

where $\Pi = \lambda_1\mathbf{w_1} + \lambda_2\mathbf{w_2} + \ldots + \lambda_{d+1}\mathbf{w_{d+1}} : 0 \le \lambda_1, \lambda_2, \ldots, \lambda_{d+1} < 1$. This parallelepiped is bounded, so the attached generating function σ_Π is a Laurent polynomial in $z_1, z_2, \ldots, z_{d+1}$.

We claim that the z_{d+1}-degree of σ_Π is at most d. In fact, since the x_{d+1}-coordinate of each $\mathbf{w}_k$ is 1, the x_{d+1}-coordinate of a point in Π is $\lambda_1 + \lambda_2 + \ldots + \lambda_{d+1}$ for some $0 \le \lambda_1, \lambda_2, \ldots, \lambda_{d+1} < 1$. But then $\lambda_1 + \lambda_2 + \ldots + \lambda_{d+1} < d + 1$, so if this sum is an integer it is at most d, which implies that the z_{d+1}-degree of $\sigma_\Pi(z_1, z_2, \ldots, z_{d+1})$ is at most d. Consequently, $\sigma_\Pi(1, 1, \ldots, 1, z_{d+1})$ is a polynomial in z_{d+1} of degree at most d. The evaluation $\sigma_\Pi(1, 1, \ldots, 1, z_{d+1})$ of this polynomial at $z_{d+1} = 1$ is not zero, because $\sigma_\Pi(1, 1, \ldots, 1, z_{d+1}) = \#(\Pi \cap \mathbb{Z}^{d+1})$ and the origin is a lattice point in Π.

Finally, if we specialize $z^{\mathbf{w}_k}$ to $z_1 = z_2 = \cdots = z_d = 1$, we obtain z_{d+1}^1, so that

$$\sigma_{\text{cone}(\Delta)}(1, 1, \ldots, 1, z_{d+1}) = \frac{\sigma_\Pi(1, 1, \ldots, 1, z_{d+1})}{(1-z_{d+1})^{d+1}}.$$

The LHS is $\text{Ehr}_\Delta(z_{d+1}) = 1 + \sum_{t\ge1} L_\Delta(t) z_{d+1}^t$ by Lemma 7.3. □

Now that we have a proof of Ehrhart's theorem in our toolbox, we can put the theorem to work. But first, let us introduce a useful result that relates the total number of lattice points in a polytope to the number of interior lattice points in it.

Theorem 7.4 (Ehrhart-Macdonald reciprocity)**.** Let $\mathcal{P}$ be a convex lattice polytope in $\mathbb{R}^d$ with positive volume and Ehrhart polynomial $L_\mathcal{P}$. Then

$$L_{\mathcal{P}^\circ}(t) = (-1)^{dim\mathcal{P}} L_\mathcal{P}(-t).$$

What's nice about Theorem 7.4 is that it gives us a counting function for just the interior lattice points of $\mathcal{P}$, namely $L_{\mathcal{P}^\circ}(t)$, from $L_\mathcal{P}(t)$ evaluated at negative integers up to a sign. The proof of Theorem 7.4 is beyond the scope of this paper, but we encourage the reader to look at Chapter 4 of [BR07] for it.

Remark 7.5. As it turns out, Theorem 7 is only one version of Ehrhart's theorem. There is an extension of the theorem to rational polytopes in which $L_\mathcal{P}(t)$ is a quasipolynomial. For more information on this topic, look at Chapter 3.7 of [BR07].

Example 7.6 (Reeve's tetrahedron)**.** Let $\mathcal{T}_h$ be the tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), and $(1, 1, h)$ where h is a positive integer (see Figure 7). We want to determine the Ehrhart polynomial of $\mathcal{T}_h$.

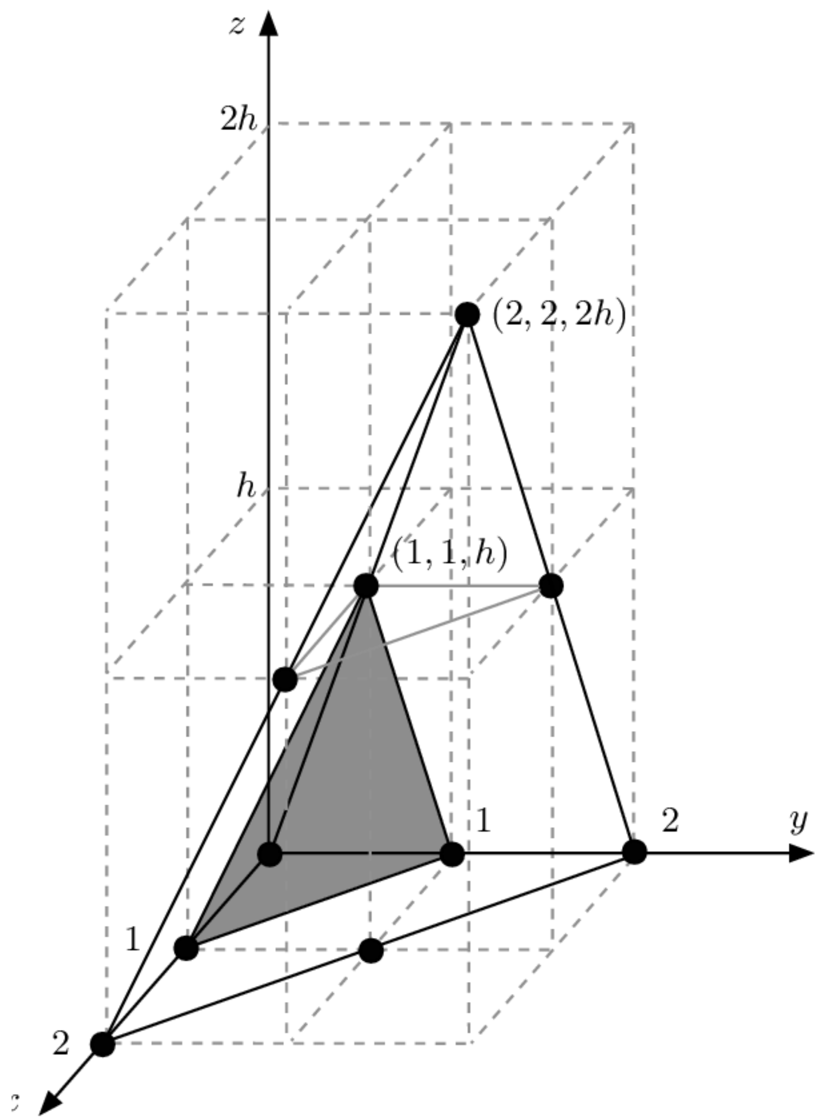

Figure 7. Reeve's tetrahedron $\mathcal{T}_h$ (and $2\mathcal{T}_h$).

We introduce a new tool, polynomial interpolation. In the case of an Ehrhart polynomial $L_{\mathcal{P}}(t)$, the interpolation equation is

$$\begin{pmatrix} L_{\mathcal{P}}(x_1) - 1 \\ L_{\mathcal{P}}(x_2) - 1 \\ \vdots \\ L_{\mathcal{P}}(x_d) - 1 \end{pmatrix} = \begin{pmatrix} x_1^d & x_1^{d-1} & \dots & x_1 \\ x_2^d & x_2^{d-1} & \dots & x_2 \\ \vdots & \vdots & \dots & \vdots \\ x_d^d & x_d^{d-1} & \dots & x_d \end{pmatrix} \begin{pmatrix} c_d \\ c_{d-1} \\ \vdots \\ c_1 \end{pmatrix}$$

Plugging in various points from the tetrahedron into our interpolation formula, we deduce the following:

$$4 = L_{\mathcal{T}_h}(1) = \mathrm{vol}(\mathcal{T}_h) + c_2 + c_1 + 1$$

$$h + 9 = L_{\mathcal{T}_h}(2) = \mathrm{vol}(\mathcal{T}_h) \cdot 2^3 + c_2 \cdot 2^2 + c_1 \cdot 2 + 1.$$

Using the volume formula for a pyramid, we know that

$$\mathrm{vol}(\mathcal{T}_h) = \frac{1}{3}(\text{base area})\ (\text{height}) = \frac{h}{6}.$$

So $h+1 = h+2c_2-1$, which gives us $c_2 = 1$ and $c_1 = 2 - \frac{h}{6}$. Therefore

$$L_{\mathcal{T}_h}(t) = \frac{h}{6}t^3 + t^2 + (2 - \frac{h}{6})t + 1.$$

□

8. Interpreting Coefficients

It turns out that the coefficients of Ehrhart polynomials encode some very important information. We aim to decode some of them here.

Let's start by uncovering the final coefficient of an Ehrhart polynomial. To do this, we employ the use of an Ehrhart series.

Lemma 8.1. Suppose $\mathcal{P}$ is an integral convex d-polytope with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d z^d + h_{d-1}z^{d-1} + \ldots + h_0}{(1-z)^{d+1}}$$

Then $h_0 = 1$.

The motivation behind this lemma requires some further knowledge from the proof of Stanley's nonnegativity theorem. We advise you to look at pg. 66 of [BR07] for this proof.

Lemma 8.2. Suppose $\mathcal{P}$ is an integral convex d-polytope with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d z^d + h_{d-1}z^{d-1} + \ldots + 1}{(1-z)^{d+1}}$$

Then

$$L_{\mathcal{P}}(t) = \binom{t+d}{d} + h_1\binom{t+d-1}{d} + \ldots + \binom{t+1}{d} + \binom{td}{d}.$$

Proof. Expand into a binomial series:

$$\begin{aligned}
\mathrm{Ehr}_{\mathcal{P}}(z) &= \frac{h_d z^d + h_{d-1}z^{d-1} + \ldots + 1}{(1-z)^{d+1}} \\
&= (h_d z^d + h_{d-1}z^{d-1} + \ldots + h_1 z + 1)\sum_{t\geq 0}\binom{t+d}{d}z^t \\
&= h_d\sum_{t\geq 0}\binom{t+d}{d}z^{t+d} + h_{d-1}\sum_{t\geq 0}\binom{t+d}{d}z^{t+d-1} + \ldots + h_1\sum_{t\geq 0}\binom{t+d}{d}z^{t+1} + \sum_{t\geq 0}\binom{t+d}{d}z^t \\
&= h_d\sum_{t\geq d}\binom{t}{d}z^t + h_{d-1}\sum_{t\geq d-1}\binom{t+1}{d}z^t + \ldots + h_1\sum_{t\geq 1}\binom{t+d-1}{d}z^t + \sum_{t\geq 0}\binom{t+d}{d}z^t.
\end{aligned}$$

In all of the infinite sums on the RHS, we can start the index t with 0 without changing the sums, by the definition of the binomial coefficient. Hence

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \sum_{t\geq 0}\left(h_d\binom{t}{d} + h_{d-1}\binom{t+1}{d} + \ldots + h_1\binom{t+d-1}{d} + \binom{t+d-1}{d} + \binom{t+d}{d}\right)z^t.$$

so

$$L_{\mathcal{P}}(t) = \binom{t+d}{d} + h_1\binom{t+d-1}{d} + \ldots + \binom{t+1}{d} + \binom{td}{d}.$$

□

Now we can utilize Lemma 8.2 to reveal the constant coefficient of an Ehrhart polynomial.

Lemma 8.3. If $\mathcal{P}$ is an integral convex d-polytope, then the constant term of the Ehrhart polynomial $L_{\mathcal{P}}(t)$ is 1.

Proof. Use the expansion of 8.2. The constant term is

$$L_{\mathcal{P}}(0) = \binom{d}{d} + h_1\binom{d-1}{d} + \ldots + h_{d-1}\binom{1}{d} + h_d\binom{0}{d} = \binom{d}{d} = 1.$$

□

Now we turn our attention towards the leading coefficient of $L_{\mathcal{P}}(t)$. Uncovering the leading coefficient of an Ehrhart polynomial first merits a discussion about the difference between discrete volume and continuous volume. Given a geometric object $S \subset \mathbb{R}^d$, its (continuous) *volume*, $\mathrm{vol}(S) := \int_S dx$, can be computed by approximating S as the sum of many small d-dimensional boxes. If we have boxes with side length $\frac{1}{t}$, then they each have volume $\frac{1}{t^d}$. We may think of these boxes as filling the space between lattice points in the lattice $(\frac{1}{t}\mathbb{Z})^d$. Now we have a nice setup for volume computation by counting boxes, or equivalently, lattice points in $(\frac{1}{t}\mathbb{Z})^d$.

Lemma 8.4. Suppose $S \subset \mathbb{R}^d$ is d-dimensional. Then

$$\mathrm{vol}(S) = \lim_{t\to\infty} \frac{1}{t^d} \cdot \#(tS \cap \mathbb{Z}^d)$$

Here S must be d-dimensional, because otherwise $\mathrm{vol}(S) = 0$ by our current definition. As the reader might guess, it turns out that we need not use this limit definition to compute $\mathrm{vol}(P)$ for an d-polytope $\mathcal{P}$. Instead, we can simply compute the coefficients of $\mathcal{P}$'s Ehrhart polynomial.

Proposition 8.5. Suppose $\mathcal{P} \subset \mathbb{R}^d$ is an integral convex d-polytope with Ehrhart polynomial $L_{\mathcal{P}}(t) = c_d t^d + c_{d-1}t^{d-1} + \ldots + c_0$. Then $c_d = \mathrm{vol}(\mathcal{P})$.

Proof. By Lemma 8.4,

$$\mathrm{vol}(\mathcal{P}) = \lim_{t\to\infty} \frac{c_d t^d + c_{d-1}t^{d-1} + \ldots + c_0}{t^d} = c_d.$$

□

Not only is this result very useful, it's also very elegant! It should not come as a total surprise, though, as the lattice points in some object should grow roughly like the continuous volume of that object as we enlarge it. What's most interesting is that in the process of computing an Ehrhart polynomial, we're getting continuous data from discrete data.

We can extend this fact even further to the Ehrhart series of a polytope.

Corollary 8.6. Suppose $\mathcal{P} \subset \mathbb{R}^d$ is an integral convex d-polytope, and

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d z^d + h_{d-1}z^{d-1} + \ldots + h_1 z + 1}{(1-z)^{d+1}}.$$

Then

$$\mathrm{vol}(\mathcal{P}) = \frac{h_d + h_{d-1} + \ldots + h_1 + 1}{d!}$$

.

Proof. Use the expansion of Lemma 8.2. The leading coefficient is

$$\mathrm{vol}(\mathcal{P}) = \frac{h_d + h_{d-1} + \ldots + h_1 + 1}{d!}.$$

□

Finally, we will briefly examine the second leading coefficient of an Ehrhart polynomial.

Theorem 8.7. Suppose $L_{\mathcal{P}}(t) = c_d t^d + c_{d-1} t^{d-1} + \ldots + 1$ is the Ehrhart polynomial of an integral polytope $\mathcal{P}$. Then

$$c_{d-1} = \frac{1}{2} \sum_{\mathcal{F}\text{a facet of}\mathcal{P}} \mathrm{vol}(\mathcal{F}).$$

This theorem establishes a relationship between the second leading coefficient c_{d-1} of the Ehrhart polynomial of the d-polytope $\mathcal{P}$ and the leading coefficients of the Ehrhart polynomials of the facets of $\mathcal{P}$. The proof of this theorem is beyond the scope of this paper as it uses Dehn-Somerville relations. For a proof, read Chapter 5 of [BR07].

Remark 8.8. The reader might wonder if the middle coefficients of Ehrhart polynomials give rise to nice expressions as well. These coefficients are much more mysterious and are an area of active research.

To conclude the paper, we will use our newfound facts about the coefficients of Ehrhart polynomials to provide an elegant proof of Pick's theorem (Theorem 4.1).

Example 8.9. Consider an integral convex polytope $\mathcal{P} \subset \mathbb{R}^2$. The Ehrhart polynomial for $\mathcal{P}$ is of the form

$$L_{\mathcal{P}}(t) = Vt^2 + at + 1$$

where $a \in \mathbb{Q}$ and V is $\mathrm{vol}(\mathcal{P})$, which is actually the area of the polytope in this case. Plugging in $t = 1$ and $t = -1$ yields

$$L_{\mathcal{P}}(t) = V + a + 1 = I + B$$

$$L_{\mathcal{P}}(t) = V - a + 1 = I$$

where I is the number of $\mathcal{P}$'s interior points and B is the number of its boundary points. Note that in the first case we're simply calculating the number of lattice points in the non-dilated polytope, which is the sum of the number of interior points and the number of boundary points. In the second case, we lose the boundary points because of the Ehrhart-Macdonald reciprocity law. Eliminating a and rearranging, we recover Pick's theorem.

$$V = I + \frac{B}{2} - 1$$

Observe that if we instead solve for a, we find that

$$a = V + 1 - I = \frac{B}{2}$$

as expected by Theorem 8.7. □

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