CATALAN OBJECTS

SIDHARTH SHARMA

Abstract. In this paper we shall first discuss Catalan Objects and what they are, in addition to different examples of Catalan Objects and what they describe.

1. Introduction

We first define Catalan Numbers and the recurrences they satisfy.

Definition 1.1. *Catalan Numbers* follow the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \prod_{k=2}^{n} \frac{n+k}{k}.$$

The first couple Catalan Numbers are $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862,$ according to OEIS, when $n = 0, 1, 2, \ldots$. As we can see, the Catalan Numbers grow rather quickly. They also satisfy a special recurrence:

Theorem 1.2. *The Catalan Numbers follow the recurrence*

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k},$$

where the initial condition is $C_0 = 1$.

We will prove this theorem after discussing Dyck Paths.

Corollary 1.3. *Catalan Numbers also satisfy*

$$C_{n+1} = \frac{2(2n+1)}{n+2} C_n.$$

Proof. This follows quite simply from the definition of the Catalan Numbers:

$$C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1} = \frac{(2n+2)(2n+1)}{(n+2)(n+1)^2}\binom{2n}{n} = \frac{2(2n+1)}{(n+2)(n+1)}\binom{2n}{n} = \frac{2(2n+1)}{n+2} C_n,$$

as desired. ■

We will now talk about Dyck Paths.

2. Dyck Paths

Say we have a Cartesian plane, and we want to stay under the line $y = x$ to get to the point (n, n). How many ways are there to do that?

Definition 2.1. Paths that go from $(0, 0)$ to (n, n) and never go over the line $y = x$ are called *Dyck Paths.*

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Remark 2.2. Dyck Paths can also be expressed as ”mountain ranges”, where each step in the x direction is an ”/” and each step in the y direction is a ”\”.

Example. Let our Dyck Path be from $(0, 0)$ to $(2, 2)$, following $RRUU$, where R is a step to the right along the x-axis, and U is a step up along the y-axis. Then our ”mountain range” diagram would be:

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As you can see the ”mountain” never goes under the line $y = 0$, and each / is a right, and each \ is a up.

Now a theorem:

Theorem 2.3. *The number of Dyck paths from* $(0, 0)$ *to* (n, n) *is*

$$\frac{1}{n+1}\binom{2n}{n}$$

Proof. We begin by observing that $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$. This crucial to the proof. We go about this using Complementary Counting. There are $\binom{2n}{n}$ paths without any restrictions on passing the line $y = x$. Now, let us count the bad paths. Let us say we have a bad path, which goes above the line $y = x$ at some point $(k, k + 1)$. Before going above the line, it is at the point (k, k), which is on the line. At that point, there have been k east steps, and $k + 1$ north steps. Thus, there are $n - k$ east steps remaining, and $n - k - 1$ north steps remaining. Now, we make every north step after that an east step, and every east step a north step. Thus, we end up taking $k + (n - k - 1)$ east steps, and $k + 1 + n - k = n + 1$ north steps, to arrive at the point $(n - 1, n + 1)$. Thus, there are then $\binom{2n}{n-1}$ bad paths, so subtracting we get $\binom{2n}{n} - \binom{2n}{n-1}$. ∎

Now, with Dyck Paths in our toolkit, let's prove the recurrence.

3. Proving the Catalan Recurrence

This proof uses the ”mountain ranges” definition of Dyck Paths.

Proof. We know that we need to make an expression for C_{n+1}, which counts Dyck paths of length $2(n + 1)$. We let $2(k + 1)$ be the first time where our path hits the line, where $0 <= k <= n$. Because of this division, the path becomes two parts: that path to the right, which has length $2(n - k) = C_{n-k}$, and the part to the left. The part to the left can be done in C_k ways, so iterating over all possible k, we get

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

. If our path doesn't hit the x-axis, then it must hit the axis at the end, where the recurrence holds. ∎

This recurrence allows us to compute more values of C_{n+1}, as well as prove that other situations are equal to the Catalan Numbers.

4. Dyck Language

We first define what *well-balanced* is:

Definition 4.1. Let $\alpha_1 = \{(,)\}$ be an alphabet consisting of "(" and ")". Given a word $a \in \alpha_1$, we say that a is *well-balanced* if

- $D_1(a) = 0$, and
- $D_1(b) \geq 0$, where b is an arbitrary prefix of a

where $D_1(w)$ is the number of occurrences of a left parentheses minus the amount of occurrences of the right parentheses in our word w.

Example. The word ()((())) is well-balanced, but)(() is not. Similarly, ()((()) is not a well-balanced word.

Using this, we define the Dyck Language:

Definition 4.2. The *Dyck Language* over α_1 is the set of well balanced words over α_1.

Remark 4.3. The subscript indicates the amount of types of parenthesis.

Question 4.4. *How many well-balanced words are there of length* $2n$*?*

Answer. We let each "(" be an "/", and each ")" be an "\". If there are more right parentheses than left, then our mountain range goes below $y = 0$, and if there is an equal amount, it intersects $y = 0$. For example, the word "(()())" is

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/    \
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Alternatively, each "(" could be a step in the x-axis, and each ")" could be a step in the y-axis. We are then trying to go to (n, n). These are Dyck Paths, so the amount of ways is C_n. ■

5. Parenthesizing

Catalan Numbers also relate to Parenthesizing.

Question 5.1. *How many ways are there to group* n *factors with parenthesis?*

We let P_n be the amount of ways to group n factors with parenthesis. We then let $P_1 = 1$.

Then, $P_2 = 1$, because if we let our two factors be a and b, then we can only have (ab). Then, $P_3 = 2$, because if we have a, b, c as our factors, then we can only have $(ab)c$ or $a(bc)$. Going on, we get that $P_4 = 5$. To find P_5, we notice that the parenthesizing has to be of the form

$$b_1(b_4), (b_2)(b_3), (b_3)(b_2), (b_4)b_1,$$

where b_n is a product of n factors. Then, we see how many ways there are to do this. There are then $P_1 \cdot P_4 = 5$ ways for the first one, and going on, and adding them up, we get 14 ways. In general, we have that

$$P_n = P_1 P_{n-1} + P_2 P_{n-2} + \ldots P_{n-2} P_2 + P_{n-1} P_1 = \sum_{k=1}^{n-1} P_k P_{n-k},$$

so the amount of ways to group n factors with parenthesis is the $n-1$th Catalan Number.

6. Binary Trees

We can find that binary trees are also related to Catalan Numbers.

Definition 6.1. A *binary tree* is an undirected graph in which each node has a maximum of 2 children.

Remark 6.2. When a node has 2 children, the children are referred to as the right node and left node.

Definition 6.3. We call a node *internal* if the node has 2 children.

Definition 6.4. We call a binary tree *rooted* if it contains a root, and each node has at most 2 children.

We now have a question:

Question 6.5. *How many rooted binary trees are there with n internal nodes?*

Binary trees may not seem to relate to Catalan Numbers, but they do:

Answer. We let B_n be the number of rooted binary trees with n internal nodes. Now, $B_0 = B_1 = 1$. Now, when $n \geq 0$, let there be a binary tree with $n+1$ nodes. There are k internal nodes on the right side of the root node, and $n-k$ internal noes on the right, since the root node is also internal. So, summing over all possible k, we have

$$B_{n+1} = \sum_{k=0}^{n} B_k B_{n-k}.$$

We recognize this recurrence as the one for Catalan Numbers, so this means $B_{n+1} = C_{n+1}$, so $B_n = C_n$. ■

7. Triangulations

It seems odd, but Catalan Numbers are related to Triangulations!

Theorem 7.1. *The number of triangulations of an $n+2$-gon into n triangles is equal to C_n.*

Proof. We do some examples, and we see that when we draw a line, we are left with another polygon, which can be triangulated. This reminds us of the Catalan Recurrence. Let us say our polygon has $n+2$ sides. We first need to call the vertices of the polygon $v_1, v_2, v_3, v_4, \ldots v_n, v_{n+1}, v_{n+2}$. When we have to start triangulation, let's say our vertices of the first triangle are $v_a, v_a + 1, v_k$. We iterate over k, to see that it becomes the Catalan Recurrence. ■

There are many many Catalan Objects that were not covered in this paper. To find out more, see [Cat] and [Sta].

References

[Cat] Catalan objects. `https://www.math.uwaterloo.ca/~nsolsonh/blog/math/catalan-objects.html`.
[Sta] `https://math.mit.edu/~rstan/ec/catalan.pdf`.