GEOMETRY OF POSETS

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1. Introduction

We'll begin by defining what posets are.

Definition 1.1. A *poset* (partially ordered set) is a set P and binary relation $\leq$ such that all $x, y, z \in P$ exhibit the following properties:

(1) Reflexivity: $\forall\ x \in P$, $x \leq x$.
(2) Transitivity: If $x \leq y$ and $y \leq z$ then $x \leq z$.
(3) Antisymmetry: If $x \leq y$ and $y \leq x$ then $x = y$.

Elements that satisfy the conditions of the binary relation are said to be *comparable*. In contrast, elements can also be *incomparable* if either $x \leq y$ or $y \leq x$. Hence, when defining a poset, it is necessary to clarify the exact nature of the relationship between the elements of the set. The following are a few examples of common posets.

Example. Let A be a set. Then, $(P(A), \subseteq)$ is a poset. In other words, the elements of the poset are all the possible subsets of A, and two elements are comparable if one is the subset of the other. This poset is called the Boolean lattice (B_n) of order n (where $B_n = |A|$). B_3, for example, is $((1,2,3),(1,2),(1,3),(2,3),(1),(2),(3)(\emptyset))$.

Example. Let n be a natural number, and let D_n be the set of divisors of n. Then, $(D_n, |)$ is a poset. Which is to say, two elements of the poset are comparable if one divides the other. More specifically, D_{18} is $(1,2,3,6,9,18)$.

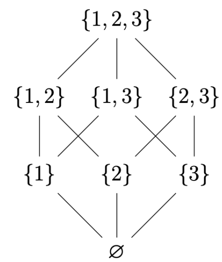
Posets can also be represented graphically, through the utilization of *Hasse diagrams*. In this way, relationships between the elements of the poset can be more easily seen and analyzed.

Definition 1.2. A *Hasse diagram* is a graphical representation of a poset with an implied upward orientation. Points are placed in representation of the elements, and lines are drawn to signify a relation. The following rules must be adhered to:

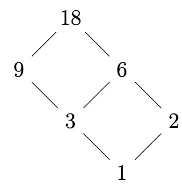
(1) Given a poset P, if $x, y \in P$ satisfy $x \leq y$, then the point representing x must be drawn lower than the point representing y.
(2) If points x and y reside on the same 'level' of the diagram, they are necessarily incomparable. However, this doesn't imply the converse.

It is important to note that loops and implicit relations need not be drawn, as partial orders are inherently reflexive and transitive. The following examples are Hasse diagrams of the posets given above.

Example. B_3



Example. D_{18}



2. Simplicial Complexes

In this section, we will cover their fundamentals; we'll give a bit of background on the basic analytic geometry of Euclidean space, and finally define exactly what simplicial complexes are.

Definition 2.1. A set of points, $(a_0, ..., a_n)$, is said to be *geometrically independent* if—for any real scalars t_i—the equations

$$\sum_{i=0}^{n} t_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} t_i a_i = 0$$

necessarily imply that $t_0 = t_1 = ... = t_n = 0$.

Theorem 1. In general, the set of points $(a_0, ..., a_n)$ is only geometrically independent if and only if the vectors

$$a_1 - a_0, a_2 - a_0, \cdots, a_{n-1} - a_0, a_n - a_0$$

are linearly independent.

Proof. Suffice to say, it's most appropriate to begin with a definition clarifying exactly what *linear independence* means. A set of vectors $[v_0, ..., v_n]$ is linearly independent if the following vector equation:

$$x_0 v_0 + x_1 v_1 + \cdots + x_{n-1} v_{n-1} + x_n v_n$$

has only the trivial solution $x_0 = x_1 = ... = x_{n-1} = x_n = 0$. Otherwise, the vector set is linearly dependent.

In other words, a linearly independent set of vectors is one in which no vector can be formed with a linear combination of any number of the other vectors.

Applying this definition to the theorem statement, lets suppose that the displacement vectors $a_1 - a_0, a_2 - a_0, ..., a_{n-1} - a_0, a_n - a_0$ are linearly independent vectors of points $a_0, ...a_n$. Then, let $t_0, t_1, ...t_n$ be real numbers satisfying the following equations.

$$\sum_{i=0}^{n} t_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} t_i a_i = 0$$

This, then implies that $t_0 = -\sum_{i=1}^{n} t_i$ and hence, the following equation can be derived:

$$0 = \sum_{i=0}^{n} t_i a_i = t_0 a_0 + \sum_{i=1}^{n} t_i a_i = \sum_{i=1}^{n} t_i (a_i - a_0)$$

In utilizing the definition of linear independence of the vectors $a_i - a_0$, we know that

$$t_1 = t_2 = ... = t_{n-1} = t_n = 0.$$

However, t_0 is also equivalent to 0, as we previously found that $t_0 = -\sum_{i=1}^{n} t_i$. Thus, since all t_i are necessarily equivalent to 0, the points $a_0, ...a_n$ are geometrically independent. □

Definition 2.2. The span $(a_0, ..., a_n)$ forms an *n-plane*, which consists of all points x of $\mathbb{R}^n$ such that

$$x = \sum_{i=0}^{n} t_i a_i$$

in which $\sum t_i = 1$. In order to further clarify, we will present an elementary theorem elucidating the nature of such *n-planes*.

Theorem 2. If $(a_0, ..., a_n)$ is geometrically independent, and if w lies outside the plane that these points span, then $(w, a_0, ..., a_n)$ is geometrically independent.

Proof. A span of a group of points $a_0, ..., a_n$ is essentially the set of all linear combinations of the vectors $a_1 - a_0, a_2 - a_0, ..., a_{n-1} - a_0, a_n - a_0$. This span of a certain group of vectors is always subspace of $\mathbb{R}^n$.
Hence, if w lies outside the span of these points, this implies:

$$t_{n+1}(w - a_0) + \sum_{i=0}^{n} t_i (a_i - a_0) = 0 \quad \text{where} \quad \sum_{i=0}^{n+1} t_i = 0$$

Which is to say, $w - a_0$ is linearly independent from $a_i - a_0$. Because of Theorem 1, we know linear independence implies geometric independence, and hence $(w, a_0, ..., a_n)$ is geometrically independent.

□

Definition 2.3. Given that $(a_0, ..., a_n)$ is a geometrically independent set in $\mathbb{R}^n$, we can define the *n-simplex* σ spanned by $a_0, ..., a_n$ in which $t_i \geq 0$ for all i and

$$x = \sum_{i=0}^{n} t_i a_i \quad \text{where} \quad \sum_{i=0}^{n} t_i = 1.$$

The numbers t_i are determined uniquely by the value of x, and as such, they are called the *barycentric coordinates*, $t_i(x)$, of the point x.

The points $a_0, ..., a_n$ that span σ are called the *vertices* of the simplex. Any simplex spanned by the subset of $(a_0, ..., a_n)$ is called a *face* of σ. The face spanned by the subset $(a_0, ..., a_{i-1}, a_{i+1}, ..., a_n)$ is known as the face *opposite* of a_i. The faces of σ not including itself are called the *proper faces* of σ; the union of these faces is called the *boundary* of σ and is denoted as Bd σ. The *interior* of σ, or *open simplex*, is defined as Int $\sigma = \sigma -$ Bd σ.

Theorem 3. An *n-simplex* is a figure in $\mathbb{R}^n$ such that it is geometrically similar to the standard N-simplex in the respective dimension.

Proof. We can begin by visualizing simplices in low dimensions. The *0-simplex* is just a point. Intuitively, a *1-simplex* consists of a_0, a_1 and all points on the line segment conjoining a_0 and a_1. This is represented by the equation

$$x = t_0 a_0 + (1 - t_0) a_1$$

where 0 $\leq t_0$ $\leq$1. In much the same way, the *2-simplex* is spanned by a_0, a_1, and a_2 and equals the triangle having these three points as its vertices. Its equation is as follows:

$$x = \sum_{i=0}^{2} t_i a_i = t_0 a_0 + (1 - t_0)[(\frac{t_1}{\lambda})a_1 + (\frac{t_2}{\lambda})a_2]$$

where $\lambda = 1 - t_0$. The expression in brackets represents the line segment between a_1 and a_2 because $\frac{(t_1+t_2)}{\lambda} = 1$ and $t_1, t_2 \geq 0$. Here, x is a point on the line segment joining a_0 and a point on $a_1 a_2$. σ equals the union of all such line segments, and thus forms a triangle.
In the same vein, the following equation for the generalization can be derived:

$$x = \sum_{i=0}^{n} t_i a_i = t_0 a_0 + \Big(t_1 a_1 + \Big(t_2 a_2 + \cdots + (1 - (t_0 + t_1 + \cdots + t_{n-2}))(\frac{t_{n-1}}{\lambda} a_{n-1} + \frac{t_n}{\lambda} a_n)\Big)\Big)$$

where $\lambda = (1 - (t_0 + t_1 + ... + t_{n-2}))$. Here, all points x lie on the face connecting a_0 to the N-1 simplex formed by $(a_1, ..., a_n)$. Finally, because σ is, once again, a union of all such faces, it takes the form of the standard geometrical N-simplex. □

The following theorems concern the fundamental properties of simplices.

Theorem 4. The barycentric coordinates, $t_i(x)$, of x with respect to $(a_0, ..., a_n)$ are continuous functions of x.

Proof. Because we already know that the values of the barycentric coordinates $(t_i(x))$ are uniquely determined by x, these coordinates are indeed a function of x. Now, simply proving continuity remains.
We can define the $n - plane$ P to be the set of points spanned by all $z = \sum_{i=0}^{n} \lambda_i a_i$ such that $\lambda \in \mathbb{R}$. This is essentially a superset of the simplex σ (which is defined by having the extra condition that $\sum_{i=0}^{n} t_i = 1$.
Thus t_i is continuous over $\mathbb{R}$ as $P \to \mathbb{R}$. Continuity over σ naturally follows, as a restriction of a continuous map is in turn itself continuous. □

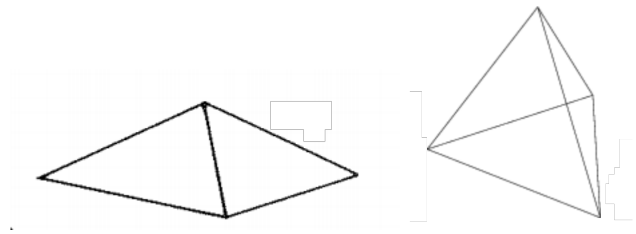
Theorem 5. σ is equivalent to the union of all line segments joining a_0 to the points on the simplex spanned by $(a_0, ..., a_n)$. Two such line segments only intersect at the point a_0.

Proof. Given the definition of a simplex, we know that it consists of all points $x = \sum_{i=0}^{n} s_i a_i$ where $s_i \geq 0$ and $\sum_{i=0}^{n} s_i = 1$. From theorem 3, we know that the line segment $t_0 a_0 + (1 - t_0) \sum_{i=1}^{n} s_i a_i$, where $0 \leq t_0 \leq 1$, is also contained in the simplex σ.
Verifying the first part of the proof, given a point $\sum_{i=0}^{n} t_i a_i$ in σ with $t_0 \neq 1$, we can set $s_i = \frac{t_i}{1-t_0}$ for $i = 1, ..., n$. This then shows that every point in the simplex spanned by. $a_0, ..., a_n$ is in the union of the line segments. a_0 is also in the union, since it is in each line segment.
Proving the second portion of the theorem statement, a_0 is clearly a common point of intersection in the union of all line segments. Assume that there is another point of intersection y. Then, the segments must lie on the same line. However, this contradicts the notion that

each point $a_0, ..., a_n$ is geometrically. independent. Thus, the only point of intersection must be a_0. ☐

Definition 2.4. A *geometric simplicial complex* K in $\mathbb{R}^n$ is a collection of simplices in $\mathbb{R}^n$.

A couple of examples of simplicial complex in the various dimensions are given below.



(a) (b)

Theorem 6. Every simplicial complex satisfies the following two properties:

(1) Every face of a simplex of K is in K.
(2) The intersection of any two simplices of K is a face of each of them.

Proof. First, we can assume that K is a simplicial complex. From this, condition (1) is immediate, as a simplicial complex can only ever be formed out of a collection of smaller simplices. However, we have yet to prove the second. Given two simplices σ and τ, we can show that if their interiors have a point x in common, then $\sigma = \tau$.
Let $s = \sigma \cap \tau$; if s were a proper face of σ, then x would belong to Bd σ, which it does not. Therefore, $s = \sigma$. In much the same way, we can prove $s = \tau$. Therefore, the intersection of any two simplices must be a face. ☐

3. Order Complexes and Face Posets

A simplicial complex is associated to every poset.

Definition 3.1. To every poset P, one can associate a simplicial complex $\triangle(P)$, called the *order complex* of P. The vertices of $\triangle(P)$ are the elements of P and the faces of $\triangle(P)$ are the chains (i.e. the totally ordered subsets) of P.

Definition 3.2. To every simplicial complex, $\triangle$, one can associate s specific poset $P(\triangle)$ called the *face poset* of $\triangle$, which is defined as the poset of non-empty faces ordered by inclusion.

If we start with a simplicial complex $\triangle$, take its face poset, and then take the order complex $\triangle(P(\triangle))$, we get a simplicial complex known as the *first barycentric subdivision* of $\triangle$. This figure is minutely dissimilar to the original complex. However, when we atttribute a topological property to a poset, the geometric realization of the order complex of the poset also has that property.

Theorem 7. Posets and T_0 topological spaces are in a one-to-one correspondence.

Proof. A space satisfies the T_0 axiom if no two points have an exactly identical set of open neighborhoods. In other words, for any two points in the topological space $x, y \in X$, there is an open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. (Note that open sets of a topological space are simply the members of said space).

We must show that this finite topological space and a partially ordered set are, in fact, the same concept. Given a finite topological. space X, we can first note that the intersection of arbitrary open sets is open, as there are only a finite number of open sets to begin with.

Thus, for each $x \in X$, we can define the minimal open set U_x to be the intersection of all open sets in X containing x. The set of these U_x forms a basis for the space X, and we can call this basis the *minimal basis*. It is called in such a way, because any other basis B for X must contain this minimal basis.

We can define a topology on X as the topology with a basis of sets $\{y \in X | y \leq x\}$ for all $x \in X$. According to this definition, if $y \leq x$, then y is contained in every basis element that also contains x. Thus y is contained in the intersection as well. as U_x. On the other hand, if, in a finite topological space, $y \in U_x$, then $y \leq x$. Hence, according to these two definitions, $y \leq x$ if and only if $y \in U_x$.

Finally, if X is a T_0 finite topological space with $x, y \in X$, then $x \in U_y$ and $y \in U_x$ implies that x and y are the same point. Translating this to poset terminology, $x \leq y$ and $y \leq x$ implies $x = y$. Thus, all the requirements for a poset have been fulfilled and a one-to-one correspondence can be drawn. $\square$

4. References

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