NECKLACE SPLITTING

RILEY 0'SULLIVAN

1. Introduction

The necklace splitting problem is an interesting problem in topological combinatorics. One variation of this problem is splitting a necklace with n beads of d types between 2 different people using no more than d cuts. This can be proven using the Ham Sandwich theorem and a moment curve. The proof outlined in this paper was presented in [3].

The necklace splitting problem can also be solved when k people split a necklace with ka_d beads of type d. It was solved by Noga Alon in 1987 using the Hobby-Rice theorem in [4]. This problem has many applications including in VLSI circut design [7].

2. Necklace Splitting Between 2

Theorem 2.1. Every open necklace with d kinds of beads can be divided equally between two people using no more than d cuts.

This can be proved using the Ham Sandwich theorem. The original formulation is that given a slice of ham and a slice of cheese on a board there exists a straight line that divides them both into equal proportions.

Ham-Sandwich Theorem A single plane can divide 3 objects in half.

To prove the Ham Sandwich theorem the Borsuk-Ulam Theorem is needed. The proof for the Ham Sandwich theorem was presented by [6].

Borsuk-Ulam Theorem Every continuous map f :$\mathbb{R}^n$ $\to$S^n identifies a pair of antipodal points.

Proof. For each three objects there will be three planes of the same gradient that cuts each object in half. Two distances, d_1 and d_2 can be defined by the distances between the three planes. Those two distances can be written as coordinates (d_1, d_2). Antipodal points always will map to negatives of one another because they are opposites. But, by the Borsuk-Ulam Theorem there is at least one pair of antipodal points that map to the same coordinate. This point must be (0,0) because it is the only point that is a negative of itself. If there is a point at (0,0) it means there

Date: July 12, 2021.

is no distance between the three planes. Therefore the three objects can be cut in half by one plane. ■

The Ham Sandwich theorem has to be generalized in order to prove the Necklace Splitting Problem.

Theorem 2.2(Finite Ham-Sandwich)**.** Let $A_1, A_2, ..., A_d \in$R^d be disjoint finite sets in general position such that no more than d points of $A_1 \cup ... \cup A_d$ are contained in any hyperplane. Then, there exists a hyperplane h that bisects each A_i such that there are exactly $\lfloor \frac{1}{2}|A_i| \rfloor$ points from A_i, in each of the open half spaces defined by h.

To prove this problem we also need a lemma for a moment curve. This is just a way of arranging the necklace so you are able to prove the theorem.

Lemma 2.3 A moment curve γ in $\mathbb{R}^d$ is defined as $\gamma = \{($t,t^2, . . . , $t^d) \in \mathbb{R}^d$ t $\in \mathbb{R}^d$ $\}$. No hyperplane $h \subset \mathbb{R}^d$ intersects the moment curve γ in $\mathbb{R}^d$ in more than d points.

Proof. The intersection of hyperplane h defined by $a_1x_1 + a_2x_2 + \cdot\cdot\cdot + a_dx_d = b$ and γ is given by the solutions to the polynomial $a_1t + a_2t^2 + \cdot\cdot\cdot + a_dt^db = 0$, which as at most d distinct roots. ■

Using all of this it is possible to prove Theorem 2.1:

Proof. We consider beads to the put on the moment curve γ(t) = (t, t_2, . . . , t_d) in $\mathbb{R}^d$ where the kth bead is placed at γ(k) = (k, k_2, . . . , k_d). It is not hard to check that the points $\{\gamma(k)\}_{k=1,...,n}$ are in general position. A_i can be defined:

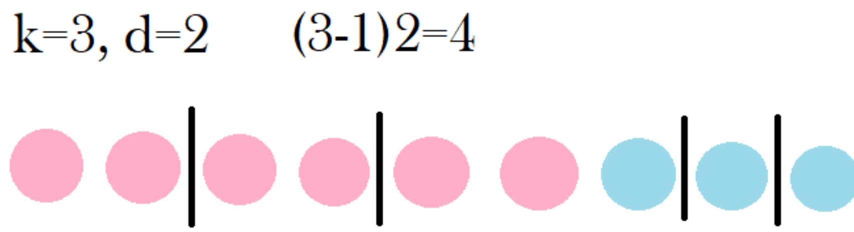
$$A_i = \{\gamma(k) \text{ — kth bead is of type i, k=1, . . ., n}\}$$

By the finite ham sandwich theorem there intersects a hyperplane h that bisects each A_i. Furthermore, by Lemma 2.3 this hyperplane meets the curve γ at most d points. ■

3. Necklace Splitting with Multiple People Background

Theorem 3.1 Given an open necklace with d different types of beads there are at most (k-1)d cuts needed to split the necklace evenly amongst k people.

This can be proven logically. The arrangement of the beads that would require the most cuts is if each type of bead was next to each other. In that case you would need to make k-1 cuts for each type of beads to evenly divide it.



This can also be proven algebraically using a generalization of the Borsuk-Ulam theorem. To prove this problem the necklace must be made continuous. This would occur by making each bead a line segment of that color. The cuts could then occur on any of the infinite points on that line. In the case that the continuous splitting cuts inside the beads, the cuts can be shifted over so they are only made within the beads. This is why the continuous version can be used to prove the discrete version of the problem.

Definition 3.2 A k-splitting of a of the necklace is a partition of the necklace into k parts, each consisting of a finite number of non overlapping intervals of beads whose union captures precisely a_i beads of color i, $1 \leq i \leq d$. The size of the k-splitting is the number of cuts that form the intervals of the splitting which is k-1.

Definition 3.3 Let I= [0, 1] be the unit interval. An interval d-coloring is a coloring of the points of I by d colors, such that for each i, $1 \leq i \leq d$, the set of points colored i is measurable.

Theorem 3.4 Every interval d-coloring has a k-splitting size of (k-1)d.

This is the same as Theorem 3.1. Theorem 3.4 also follows from the two following propositions:

Proposition 3.5 Theorem 3.4 holds for every prime k.

Proposition 3.6 The validity of Theorem 3.4 for (d, k) implies its validity for $(d,\, kl)$

Proof. To obtain a kl splitting size of $(k \times l - 1) \times d$ given an interval d-coloring. Start by using (k - 1)d cuts to form k families of intervals each capturing $\frac{1}{k}$ of the measure of each color. For each of these families, consider the interval coloring formed by placing its intervals next to each other and rescaling to total length 1. Using (l - 1)d cuts, we obtain an l-splitting of this coloring. Transforming back to the original interval coloring, this adds together to k(l- 1)d more cuts, so altogether we have (k - l)d + k(l- 1)d = (kl - 1)d cuts which form the desired kl-splitting. ■

4. Proof of Theorem 3.1

To prove theorem 3.1 we need the Hobby-Rice Theorem, a generalization of the Borsuk-Ulam theorem [2]. It can be generalized to fit the necklace splitting problem.

Theorem 4.1 Let u_1,u_2 ,..., u_d, be d continuous probability measures on the unit interval. Then it is possible to cut the interval in (k - 1)d places and partition the (k - 1)d + 1 resulting intervals into k families F_1,F_2,...,F_k such that $u_i(\cup F_j) = 1/k$ f or all $1 \leq i \leq d, 1 \leq j \leq k$. The number (k-l)d is best possible.

Given d measures u_1 ,..., u_d on the unit interval, a k-splitting of size r is a sequence of numbers $0 = y_0 \leq y_1$ . . . $\leq$y$_r \leq y_{r+1}$ and a partition of the family of r + 1 intervals F= $\{[y_i, y_{i+1}]$: $0 \leq i \leq r$ $\}$into k pairwise disjoint subfamilies F_1 ,..., F_k whose union is F, such that for each $1 \leq j \leq k$ and $1 \leq i \leq d$,

$$\mu_i(\cup F_j) = \frac{1}{k}\mu_i([0,1])$$

Theorem 3.1 as written above does not use the fact that d measures come from an interval coloring. The only requirement is that the measures are continuous and the sum of the d lengths of an interval is its length. Therefore we can write the theorem differently.

Lemma 4.2 Let m_1, m_2,...,m_d, be d continuous measures on the unit interval and suppose

$$m_1([0,\alpha]) + ... + m_d([0,\alpha]) = \alpha$$

for all $0 \leq \alpha \leq 1$ then for all $k \geq 1$ there exists a k-splitting size of (k-1)d.

Theorem 3.1 can now be proved.

Proof. Let μ_1, μ_2, . . . , μ_d be d continuous measures on the unit interval I. Suppose $\mathcal{E} \leq 0$. Define the following d measures m_1, . . . ,m_d on I. For $i \leq j \leq d$ put $m_j = \mu_j/k$ and define $m_d = (\mu_i + \mathcal{E} \times m_L)/(1+\mathcal{E}) \times k$. Put $m = m_1 +$ . . . $+$ m_d and define f: [0,1] $\rightarrow$ [0,1] by f(x)=m([0,x]). The function f is continuous, onto, and strictly increasing. Because of this its inverse f^{-1} is continuous and strictly increasing.

For $1 \leq j \leq d$ let $m_j^{'}$ be the measure given by $m_j^{'}$(S)=$m_j(f^{-1}$(S)). Clearly, for every $0 \leq \alpha \leq 1$

$$m_j^{'}([0,\alpha] + ... + m_d^{'}([0,\alpha]) = \sum{}^t{}_{j=1} m_j(f^{-1}[0,\alpha] = m([0, f^{-1}(\alpha)]) = \alpha$$

Therefore, by Lemma 4.2 there is a k-splitting size of (k-1)d for the measures $m_1^{'}$, $m_2^{'}$, . . . $m_d^{'}$ The function f^{-1} will carry this k-splitting into a k-splitting of the same size for the measures $m_1, m_2, ...m_d$.Let $F_1, F_2, ..., F_k$ be the k collections of intervals that form this spitting. By the definition of the $m_i^{'}$'s these collections almost form a k-splitting for the original measures $\mu_1, \mu_2, ..., \mu_d$.

$$\mu(\cup F_j) = \frac{1}{k}\mu_i([0,1])$$

for $1 \leq i<d$ $1 \leq j \leq k$

$$\mu(\cup F_j) + \mathcal{E} m_L(\cup F_j) = (1+\mathcal{E})/k$$

for $1 \leq j \leq k$ This means that

$$\frac{1}{k} - \frac{k-1}{k}\mathcal{E} \leq \mu_d(\cup F_j) \leq \frac{1+\mathcal{E}}{k}$$

for $1 \leq j \leq k$

By choosing a sequence $\mathcal{E}_I \to 0$ and obtaining a convergent subsequence of the sequence of k-splittings of size (k-1)d satisfying the two above equations for these $\mathcal{E}_I$'s we obtain a k-splitting size (k-1)d for the measures $\mu_1, ..., \mu_d$. ■

5. Conclusion

The necklace problem is a popular problem in topological combinatorics. In this paper we proved that when k=2 a necklace with d different types of beads can be split equally using no more than d cuts. This was proven using the Finite Ham Sandwich Theorem. We also proved that when $k>2$ the number of cuts is (k - 1)d using the Hobby-Rice Theorem. The problem can be used to efficiently divide different things equally, which can be especially helpful in the field of computer science.

References

[1] N. ALON AND D. B. WEST. *The Borsuk-Ulam theorem and bisection of necklaces.* Proc. Amer. Math. Soc. 1989
[2] C.R. Hobby and J.R. Rice A moment in L_1 approximation. *ZProc. Amer. Math Soc..* 1965
[3] Christopher Eur. "A Brief Glimpse of Topological Combinatorics". `http://www-cs-faculty.stanford.edu/~uno/abcde.html`
[4] Noga Alon. *Splitting Necklaces* 1987.
[5] S. N. Bhatt and C. E. Leiserson, *How to aesemble tree machines*, Proc. 14th ACM Sympos. Theor. Computing, San Francisco, Assoc. Comp. Mach., 1981, pp. 99-104.
[6] Zach Star *A surprising topological proof - Why you can always cut three objects in half with a single plane.* URL: https://www.youtube.com/watch?v=cchIr1OXc8Et=54s