Combinatorial Species

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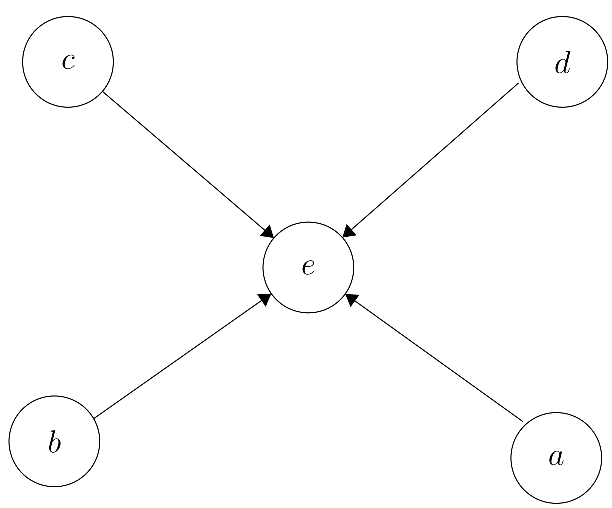
Abstract

In this paper we provide a brief introduction to combinatorial species and illustrate their use in counting problems. Combinatorial species allows us to give an algorithmic description of many counting problems which translates int generating functions. Much of the material is taken from *Combinatorial Species and Tree-like Structures* by F. Bergeron, Bergeron F, Gilbert Labelle, Pierre Leroux. This paper is a part of Combinatorics class of 2021 and serves as a resource where students can learn from.

1 Introduction

Combinatorial species provide a way of deriving generating functions of discrete structures which allow us to count these structures. Category theory provides a useful language for concepts that arise here but it's not necessary to category theory to work with cominatorial species. We begin by introducing combinatorial species and some applications.

A structure s is a construction which one performs on a set X. For example consider the set $U = \{a, b, c, d, e\}$ and let $\gamma = \{(a, e), (b, e), (c, e), (d, e)\}$. Figure below expresses structure $s = \{U, \gamma\}$ as a directed graph.



The main property will be the transport of structures along bijections. Following example illustrates this.

Example. Consider the structure in uor previous figure and the set $V = \{a_1, a_2, a_3, a_4, a_5\}$. Let $\sigma : U \to V$ be a bijection. Replace each element of U by an element of V via the bijection σ. Then the structure $t = \{V, \tau\}$ which is the directed graph is isomprphic to the graph above.

We can consider two structures equivalent if they are isomorphic or one can be obtained from other by replcaing labels. We may also consider the isomorphism type or equivalence class of all structures which are equivalent.

We can define species formally as follows

Definition 1.1. A species of structures is a rule F which produces

1. For each fiite set U a finite set $F[U]$
2. For each bijection $\sigma : U \to V$, a function $F[\sigma] : F[U] \to F[V]$.

Furthurmore the functions should satisfy

1. For all bijections $\sigma : U \to V$ and $\tau : V \to W$

$$F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$$

2. For the identity map $I : U \to U$,

$$F[I_U] = I_{F[U]}$$

The following are equivalent

1. $s \in F[U]$
2. s is an F- structure on U
3. s is a structure of species F on U

Example. Let End(U) denote the set of endofunctions i.e. $f : U \to U$. So $(\gamma, U) \in$ End(U) iff

$$\gamma \subset U \times U \text{ and } (\forall x)[x \in U \longrightarrow (\exists! y)[y \in U \,\text{and}\, (x, y) \in \gamma]]$$

Let $\sigma : U \to U$ be a bijection. Then it's easy to see that $F[\sigma] = \sigma\gamma\sigma^{-1}$. Index the elements of U and V such that $f(u_i) = v_i$. WLOG , assume that $f(u_i) = u_j$. Then $\sigma\gamma\sigma^{-1}(u_i) = v_j$.

Definition 1.2. Consider two F structure $s_1 \in F[U]$ and $s_2 \in F[V]$. A bijection $\sigma : U \to V$ is an isomorphism of s_1, s_2 if $s_2 = F[\sigma](s_1)$. We say that these structures have the same isomorphism type.

Proposition 1.3. *$F[\sigma]$ is a bijection.*

Proof. It suffices to show that $F[\sigma]$ has an inverse. σ being a bijection it has an inverse σ^{-1}. From the definition of species we have

$$F[\sigma \circ \sigma^{-1}] = F[I_U] = F[\sigma^{-1} \circ \sigma] = I_{F[U]}$$

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By set-theoretic axioms we can introduce the following species

- The species $\mathcal{A}$ of rooted trees
- The species $\mathcal{G}$ of simple graphs
- The species $\mathcal{G}^c$ of connected graphs
- the species $\mathfrak{a}$ of trees
- The species $\mathcal{D}$ of directed graphs
- The species Par of set partitions
- The species $\mathfrak{p}$ of subsets i.e., $\mathfrak{p}[U] = \{S | S \subseteq U\}$
- The species End of endofunctions
- The species Inv of involutions
- The species S of permutations
- The species $\mathcal{C}$ of cyclic permutations
- The species L of linear orders

When structure of species F is simple it can be useful to define the species by description of set $F[U]$ and transport along bijections $F[\sigma]$. the following exapmles illustrate this

- The species E, of sets is defined as $E[U] = \{U\}$
- The species ϵ, of elements is defined as $\epsilon[U] = U$
- The X , of species of characteristic singeltions is defined by setting

$$X[U] = \begin{cases} \{U\} & \text{if } |U| = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

- The species E_n where $n \in \mathbb{N}$ is defined as

$$E_n[U] = \begin{cases} \{U\} & \text{if } |U| = n \\ \emptyset & \text{otherwise} \end{cases}$$

2 Generating functions

An $F-$structure $s \in F[U]$ is referred to as labelled structure while the isomorphis class of the $F-$structure is unlabelled. Note that F is a bijection , the cardinality of $F[U]$ only depends upon the cardinality of U and not the elements of U. From here on we let $U = [|U|]$. The following three series are

1. The exponential generating function $F(x)$ for labelled eumeration
2. The ordinary generating function $\widetilde{F}(x)$ for unlabelled enumeration
3. The cycle index series $Z_F(x_1, x_2, ...)$

2.1 Exponential generating function

Definition 2.1. The exponential generating function of a species of structures F is the formal power series

$$\sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$$

where $f_n = |F([n])|$. Taking the taylor seies expansion at the origin ,

$$n![x^n] = n! \frac{d^n}{dx^n} F(x)\Big|_{x=0} = f_n$$

Example. It is easy to verify by direct enumeration the following identities

1. $L(x) = \dfrac{1}{1-x}$
2. $S(x) = \dfrac{1}{1-x}$
3. $C(x) = -\log(1-x)$
4. $E(x) = e^x$
5. $E_n(x) = \dfrac{x^n}{n!}$

2.2 Type generating function

We define an equivalence class $\sim$ on $F[n]$ for $s, t \in F[n]$ by setting

$$s \sim t \text{ iff they have the same isomorphism type}$$

s, t are in the same isomorphism type if there exists a permutation $\pi : [n] \to [n]$ such that $F[\pi](s_1) = s_2$. By definition isomorphism type of Fstructures is an equivalence class of F structures on $[n]$. Denote $T(F_n)$ to be the quotient set $F[n]/\sim$.

Definition 2.2. The type generating series of species of structure F is the formal power series

$$\tilde{F}(x) = \sum_{n=0}^{\infty} \widetilde{f}_n x^n$$

where $\widetilde{f_n} = |T(F_n)|$.

Example. By direct enumeration it is easy to yield the following generating functions

1. $\widetilde{L}(x) = \dfrac{1}{1-x}$
2. $\widetilde{S}(x) = \prod_{k=1}^{n} \dfrac{1}{1-x^k}$
3. $\widetilde{C}(x) = \dfrac{x}{1-x}$
4. $\widetilde{E}(x) = \dfrac{1}{1-x}$
5. $\widetilde{E_n}(x) = x^n$

2.3 Cycle index series

Definition 2.3. Let U be a finite set and σ be a permutation of U. The cycle type of the permutation σ is the list $(\sigma_1, \sigma_2, \ldots)$ where σ_i is the number of $i-$cycles in the cycle decomposition of the permutation. Let

$$\text{fix}(\sigma) = \sigma_1$$

Each permutation σ of $[n]$ induces a permutation of set $F([n])$ of F-structures of $[n]$.

Definition 2.4. The cycle index series of a species of structures F is the formal power series

$$Z_F(x_1, x_2, \ldots) = \sum_{n\geq 0} \sum_{\sigma \in S_n} \text{fix}F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} \ldots$$

where S_n is the group of permutations of $[n]$ and $\text{fix}F[\sigma]$ is the number of F-structures on $[n]$ fixed by $F[\sigma]$.

Theorem 2.5. *(Burnside's lemma) let G be a finite group that acts on a set X. For each $g \in G$ let fix(g) denote the set of elements in X that are fixed by g i.e. fix$(g) = \{x \in X | g \cdot x = x\}$. Burnside's lemma asserts the following formula for the number of orbits, denoted $|X/G|$*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |\mathit{fix}(g)|$$

Theorem 2.6. *For any species of structures F we have*

- $F(X) = Z_F(x, 0, 0, \dots)$
- $\tilde{F}(X) = Z_F(x, x^2, x^3, \dots)$

Proof. Note that

$$Z_F(x, 0, 0, \dots) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in S_n} \text{fix} F[\sigma] x^{\sigma_1} 0^{\sigma_2} \dots \right)$$

for a fixed n , $x^{\sigma_1} 0^{\sigma_2} \dots$ is not zero iff $\sigma_1 = n$ i.e. the identity permutation. Note that from the definition of species, $F[Id] = Id_{F[n]}$ and so $\text{fix} F[Id_n] = f_n$. Our sum becomes

$$\sum_{n \geq 0} \frac{1}{n!} \text{fix} F[Id_n] x^n$$

$$= \sum f_n \frac{x^n}{n!}$$

so our claim follows. Our second claim follows from Burnsides lemma by noting that $\tilde{f}_n$ is the number of orbits of S_n acting on $[n]$.

Remark 2.7. For all species F and all permutations σ of U, the cycle type $((F[\sigma])_1, (F[\sigma])_2, \dots)$ only depends upon the cycle type $(\sigma_1, \sigma_2, \dots)$ and is easily checked using functoriality of F i.e. for all sets U and V such that $\exists$ a bijection from U to V the permutation induced by σ on $F[U]$ corrosponds to the permutation induced by σ on $F[V]$. Hence, all permutations having the same cycle type contribute to the same monomial and since by next proposition the number of permutations having cycle type $(n_1, n_2, n_3, \dots)$ is given by

$$\frac{n!}{1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \dots}$$

the cycle index of any species is

$$Z_F(x_1, x_2, x_3, \dots) = \sum_{n_1 + 2n_2 + 3n_3 + \dots < \infty} \text{fix} F[n_1, n_2, n_3, \dots] \frac{x_1^{n_1} x_2^{n_2} \dots}{1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \dots}$$

Theorem 2.8. *(orbit-stabiliser theorem) If a finite group G acts on a set X, then for every $x \in X$, we have $|G| = |O(x)||Stab(x)|$ where $O(x)$ is the orbit of x and $Stab(x)$ is the stabiliser of x.*

Proposition 2.9. *Conjugating does not change cycle type i.e. for $\sigma, \tau \in S_n$, the cycle type of σ is same as that of cycle type of $\sigma \circ \tau \circ \sigma^{-1}$.*

Proof. Suppose $\eta = \sigma \circ \tau \circ \sigma^{-1}$ and $\tau(i) = j$. Now notice that

$$\eta(\sigma(i)) = \sigma \circ \tau \circ \sigma^{-1} \circ (\tau(i)) = \sigma \circ \tau(i) = \sigma(j)$$

so the cycle structure of η and σ is same. ∎

Example. Number of permutation of cycle type $(n_1, n_2, \ldots, n_k)$ is

$$\frac{n!}{1^{n_1}n_1!2^{n_2}n_2!\ldots k^{n_k}n_k!}$$

Define a group action $\cdot$ on S_n such that $\sigma \cdot \gamma = \sigma \circ \gamma \circ \sigma^{-1}$. From proposition above conjugating does not change the cycle type of a permutation and that $\sigma \circ \gamma = \sigma \circ \tau$ if and only if $\gamma = \tau$. Thus the size of orbit is

$$\frac{n!}{1^{n_1}n_1!2^{n_2}n_2!3^{n_3}n_3!\ldots n^{n_n}n_n!}$$

and we know that $|S_n| = n!$ so by orbit stabiliser theorem we get that

$$\text{fix}F[n_1, n_2, n_3, \ldots] = 1^{n_1}n_1!2^{n_2}n_2!3^{n_3}n_3!\ldots n^{n_n}n_n!$$

and the cycle index polynomial of the species S of permutations is

$$Z_S(x_1, x_2, \ldots) = \prod_{i\geq 1} \frac{1}{1 - x_i}$$

Example. As an example of theorem above consider the species S of permutations. It follows that

$$Z_S(x, 0, 0, \ldots) = \frac{1}{1-x} = S(x)$$
$$Z_S(x, x^2, x^3, \ldots) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots} = \widetilde{S}(x)$$

which can be verified from examples in previous section.

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3 Operations on species

In this section we introduce the reader to basic operations on generating functions. Given a sequence $\{a_i\}_{i=0}^{\infty}$ we define its ordinary generating function to be

$$\sum_{i\geq 1} a_i x^i$$

and it's exponential generating function to be

$$\sum_{i\geq 1} a_i \frac{x^i}{i!}$$

addition and multiplication of ordinary generating functions are as follows

$$\sum_{i\geq 1} a_i x^i + \sum_{j\geq 1} b_j x^j = \sum_{k\geq 1} (a_k + b_k) x^k$$

$$\left(\sum_{i\geq 1} a_i x^i\right)\left(\sum_{j\geq 1} b_j x^j\right) = \sum_{k\geq 1} x^k \left(\sum_{n=1}^{k} a_n b_{k-n}\right)$$

and addition and multiplication of exponential generating functions as follows

$$\sum_{i\geq 1} a_i \frac{x^i}{i!} + \sum_{j\geq 1} b_j \frac{x^j}{j!} = \sum_{k\geq 1} (a_k + b_k) \frac{x^k}{k!}$$

$$\left(\sum_{i\geq 1} a_i \frac{x^i}{i!}\right)\left(\sum_{j\geq 1} b_j \frac{x^j}{j!}\right) = \sum_{k\geq 1} \frac{x^k}{k!} \left(\sum_{m+n=k} \binom{k}{n} a_n b_m\right)$$

Note that if $f(x)$ is the generating function for $\{a_i\}_{i=0}^{\infty}$ then the generating function for $\{b_i\}_{i=0}^{\infty}$ where $b_i = \sum_{k\leq i} a_k$ is

$$\frac{f(x)}{1-x}$$

Composition $f \circ g$ of power series is defined if and only if $[x^0]g(x) = 0$ i.e. the constant term in g is zero.

3.1 Sum of species of structures

Definition 3.1. Let F, G be two species of structures. The species $F + G$ called as sum of F and G defined as a $(F + G)$ structure on U is the disjoint union of F and G structures on U. In other words, for any finite set U,

$$(F + G)[U] = F[U] + G[U] \quad ('+' \text{stands for disjoint union})$$

Furthermore for all bijections $\sigma : U \to V$

$$(F + G)[\sigma](s) = \begin{cases} F[\sigma](s) & \text{if } s \in F[U] \\ G[\sigma](s) & \text{if } s \in G[U] \end{cases}$$

The operation of addition is commutative and associative upto isomorphism. Given two species F and G, it's easy to see that the associated series of species $F+G$ satisfy

$$a)(F + G)(x) = F(x) + G(x)$$
$$b)F \tilde{+} G = \tilde{F}(x) + \tilde{G}(x)$$
$$c)Z_{F+G} = Z_F + Z_G$$

The operation of adition can be extended to summable families of species a s following

Definition 3.2. A family $(F_i)_{i\in I}$ of species of structures is said to be summable if for any finite set U, $F_i[U] = \emptyset$, except for a finite number of indices $i \in I$. The sum of summable family $(F_i)_{i\in I}$ is said to be species $\sum_{i\in I} F_i$ defined by the equalities

$$a)\ \left(\sum_{i\in I} F_i\right)[U] = \sum_{i\in I} F_i[U] = \cup_{i\in I} F_i[U] \times \{i\}$$

$$b)\ \left(\sum_{i\in I} F_i\right)[\sigma](s,i) = (F_i[\sigma](s), i)$$

3.2 Product of species of structures

As an instance of product of species consider any permutation. A permutation consists of a set of fixed points and a set of non-trivial cycles. We say that the species S of permutations is the product of the species E of sets and species Der of derangements written as $S = E\cdot \mathrm{Der}$ Formally, product of species is defined as follows

Definition 3.3. Let F and G be two species of structures. The species $F \cdot G$ called as the product of species F and G, is defined as follows: an $(F \cdot G)$ structure on U is an ordered pair $s = (f, g)$ where

1. f is an F structure on a subset $U_1 \subseteq U$
2. g is a G structure on a subset $U_2 \subseteq U$
3. U_1 and U_2 partition U i.e. $U_1 \cap U_2 = \emptyset$, $U_1 \cup U_2 = U$

The transport along bijection $\sigma : U \to V$ is carried out by setting, for each $F \cdot G$ structure $s = (f, g)$ on U ,

$$(F \cdot G)[\sigma](s) = (F[\sigma_1](f), G[\sigma_2](g))$$

where, σ_1 is the restriction of σ on U_1 and σ_2 is the restriction of σ on U_2.

The product of species is associative and commutative upto isomorphism but in general $F\cdot G$ and $G\cdot F$ are not identical. It is easy to see that if F and G are two species of structures, the series associate with $F \cdot G$ satisfy

1. $(F \cdot G)(x) = F(x)G(x)$
2. $F \tilde{\cdot} G(x) = \tilde{F}(x)\tilde{G}(x)$
3. $Z_{F\cdot G}(x_1, x_2, x_3, \ldots) = Z_F(x_1, x_2, x_3, \ldots)Z_G(x_1, x_2, x_3, \ldots)$

   Example. Using the above discussion and the fact that $S = E \cdot \mathrm{Der}$ we have

   (a) $\frac{1}{1-x} = e^x \mathrm{Der}(x)$

(b) $\prod_{k\geq 1} \frac{1}{1-x^k} = \frac{1}{1-x}\widetilde{\mathrm{Der}}(x)$

(c) $\prod_{k\geq 1} \frac{1}{1-x_k} = \exp\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots\right) \cdot Z_{\mathrm{Der}}(x_1, x_2, x_3, \ldots)$

Thus we deduce that

$$\mathrm{Der}(x) = \frac{e^{-x}}{1-x}$$

which produces the classical formula

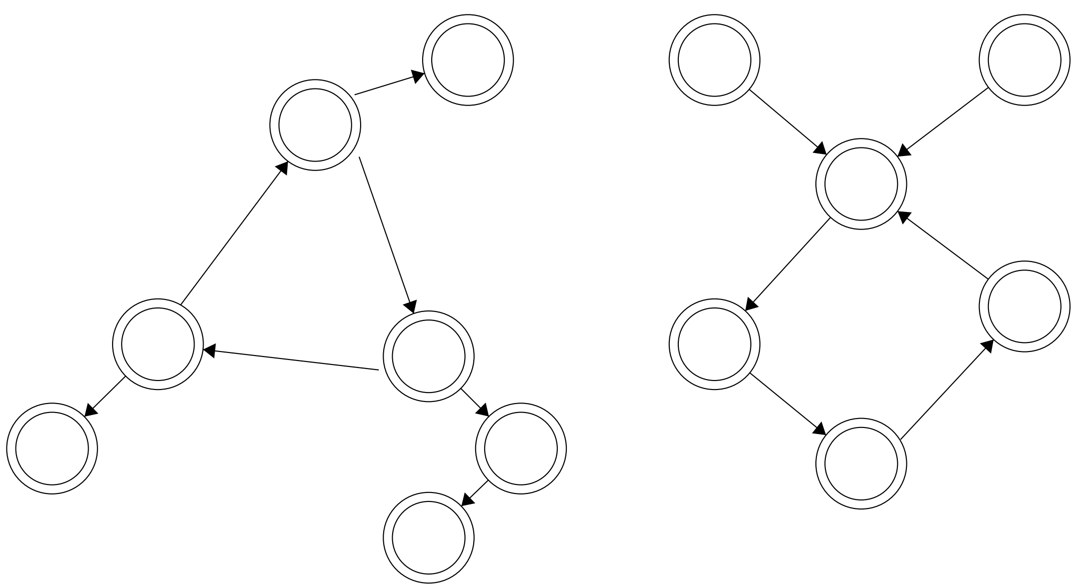
$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!}\right)$$

3.3 Substitution of species of structures

As a motivating exaample consider the speces of endofunctions. We associate a diagraph to a endofunction $\alpha : U \to U$ be setting the set U to be the set of vertices and $V = \{(u_1, u_2) | u_1, u_2 \in U, \alpha(u_1) = u_2\}$ to be the set of edges. Elements of U can now belong to two categories

1. The points for which there exists $k > 0$ such that $\alpha^k(u) = u$(recurrent points)
2. The points for which $\alpha^k(u) \neq u$ for all $k > 0$(non-recurrent points)

Below figure shows that endofunctions can be identified as permutations of rooted trees(note that we have displayed a 3-cycle and a 4-cycle of rooted trees also notice that the root is the vertex attached to the cycle which are all recurrent states).



Thus every endofunction can be identified by permutation of rooted trees or in other words by placing a S structure on a set of disjoint A structures.

Definition 3.4. Let F and G be two species of structures such that $G[\emptyset] = \emptyset$. The species $F \circ G$ also denoted as $F(G)$, called the composite of G in F , is defined as follows : an $(F \circ G)$-structure on U is a triplet $s = (\pi, \alpha, \gamma)$ where

1. π is a partition of U
2. α is an F-structure on the set of classes of π
3. $\gamma = (\gamma_p)_{p \in \pi}$, where for each class p of π, γ_p is a G structure on p.

The transport along bijection $\sigma : U \to V$ is carried out by setting , for any $F \circ G$ structure $s = (\pi, \alpha, (\gamma_p)_{p \in \pi})$ on U,

$$(F \circ G)[\sigma](s) = (\overline{\pi}, \overline{\alpha}, (\overline{\gamma_{\overline{p}}})_{\overline{p} \in \overline{\pi}}$$

where

1. $\overline{\pi}$ is a partition of V obtained by transport of π along σ
2. for each $\overline{p} = \sigma(p) \in \overline{\pi}$, the structure $\overline{\gamma_{\overline{p}}}$ is obtaied by structure γ_p by G-transport along $\sigma|_p$
3. the structure $\overline{\alpha}$ is obtained from the structure α by F-transport along the bijection $\overline{\sigma}$ induced on π by σ

also the series related to $F \circ G$ satisfy

1. $(F \circ G)(x) = F(G(x))$
2. $\widetilde{(F \circ G)}(x) = Z_F(\widetilde{G}(x), \widetilde{G}(x^2), \widetilde{G}(x^3), \ldots)$
3. $Z_{F \circ G}(x_1, x_2, x_3, \ldots) = Z_F(Z_G(x_1, x_2, x_3, \ldots), Z_G(x_2, x_4, x_6, \ldots))$

Example. From the combinatorial equation at $\text{End} = S \circ A$ we deduce that

1. $\text{End}(x) = S \circ A = \frac{1}{1 - A(x)}$
2. $\widetilde{\text{End}}(x) = \widetilde{(S \circ A)}(x) = Z_S(\widetilde{A}(x), \widetilde{A}(x^2), \widetilde{A}(x^3), \ldots)$
3. $Z_{\text{End}} = (Z_S \circ Z_A)(x_1, x_2, x_3, \ldots)$

4 Some applications

In this section we provide solutions to combinatorial problems using species.

Example. Consider the species A of rooted trees. We could generate rooted trees bi first picking a root and then attaching trees to the vertex which is picked as a root. This is exactly the definition of product of species. Thus we have

$$A(x) = xe^{A(x)}$$

We recover cayley's formula from here via lagarange inversion as

$$\left(\frac{d}{dz}\right)^{n-1} (\exp(z))^n\Big|_{z=0} = n^{n-1}$$

Thus there are n^{n-1} rooted trees on n vertices.

Example. Consider the species Par of partitions. Then any partition of a set is a set of non-empty sets. Thus if we let E_+ to be the species of non-empty subsets then we have

$$\text{Par} = E(E_+)$$

which gives us

1. $\text{Par}(x) = e^{e^x - 1}$
2. $\widetilde{\text{Par}(x)} = \prod_{k\geq 1} \frac{1}{1-x^k}$

Example. Suppose we want to count the number of set partitions of of set U into subsets with cardinality greater than some fixed k. Then the associated species is $E(E_{\geq n})$ where $E_{\geq n}$ is the species described by

$$E_{\geq n}[U] = \begin{cases} |U| & \text{if } |U| \geq n \\ \emptyset & \text{otherwise} \end{cases}$$

The generating function is given by

$$\text{Par}_{\geq k} = e^{e^x - \sum_{i=0}^{k} \frac{x^i}{i!}}$$

Example. There's also an alternative way to look at derangements. Any derangements is a set of cycles with such that each cycle has at least two elements. So derangements are equivalent to the species $E(C_{\geq 1})$ where $C_{\geq 1}$ is the species of cycles with no fixed points. Thus $C_{\geq 1}(x) = -\log(1-x) - x$ and our generating function is

$$e^{-\log(1-x)-x} = \frac{e^{-x}}{1-x}$$

Example. Let B be the species of binary trees. Then any binary tree is a pair of binary trees or a single vertex. Thus the species of binary trees satisfies

$$B = x + E_2 \circ B$$

and this translates to

$$B(x) = x + \frac{B(x)^2}{2}$$

and can be solved to

$$B(x) = 1 - \sqrt{1-2x} = \sum_{n=1}^{\infty} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)\frac{x^n}{n!}$$

Example. Suppose we want to count number of surjections from $[n]$ to $[r]$. Let $n \geq r$ otherwise the number of surjections is zero. Assume that $\sigma : [n] \to [r]$ is a surjection. Any surjection can be identified from the set $\{\sigma^{-1}(i) | i \in [r]\}$. Thus any surjection is a set consisting of r non-empty sets. This translates to the species $E_r(E_{\geq 1})$ whose generating function is

$$(e^x - 1)^r$$

and notice that

$$[x^k](e^x - 1)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (r-i)^n$$

Example. Consider the species Inv of involutions. Clearly an involution can have cycles of size 1 or 2. Thus the species Inv is isomorphic to $E(\mathrm{C}_{\leq 2})$ where $\mathrm{C}_{\leq 2}$ is the species one cycle and a two cycle whose generating function is $x + \frac{x^2}{2}$. Thus the exponential generating function of involutions is $e^{x+\frac{x^2}{2}}$. So the number of involutions on a n set is

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k)!2^k}$$

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