Q-ANALOGUES

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Abstract. For a statement X, we define a q-analogue of X as a statement X_q such that letting $q = 1$ or taking the limit as q approaches 1 of X_q results in X. In this expository paper, we aim to study several applications of q-analogues. We begin by exploring the q-analogues of binomial coefficients and the binomial theorem; for the latter, we introduce and prove both the finite and infinite versions. We then discuss two intriguing applications of q-analogues: the first is to counting lines in finite geometries, especially as it pertains to the card game SET; the second is to the mathematical modeling of juggling. Lastly, we explore several q-analogues to combinatorial numbers including Catalan numbers, Lucas numbers, Bernouilli numbers, and Narayana numbers. Next we look at Hypergeometric Series, Heine's tranformation formulas, Heine's q-analogue of Gauss summation formula and the Bailey-Daum summation formula. At the end we also give an elementary proof of Jacobi's triple product identity using q-analogues.

1. Introduction and Preliminaries

Definition 1.1. A *q-analog* of a quantity or formula P is a quantity or formula P_q, such that if we set $q = 1$ or take $\lim_{q\to 1} P_q$, we get P.

We define the q-analog of a non-negative integer n as

$$\frac{1-q^n}{1-q}.$$

Note that
$$\frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^n.$$
So, as $q \to 1$, the function gives an output of n.

We can also consider q-analogues to be a special type of generating function. q-analog of n can be defined as the generating function of the number of Ferrers boards with k squares, contained in a in a $1 \times n$ board(1 row, n columns). Another type of q-analog is the q-factorial:
$$[n]_q! = [n]_q[n-1]_q...[1]_q.$$

2. q-Binomial Coefficients and the q-Binomial theorem

We begin our discussion of q-analogues with q-analogues of binomial coefficients, known as q-binomial coefficients or Gaussian coefficients. In this section, we will first look at the recurrence relation of Gaussian coefficients. Then, we will look at a combinatorial interpretation of these q-analogues and finally we will prove the q-binomial theorem. Lets start by looking at the definition of the Gaussian coefficients.

Recall that we define the normal binomial coefficient for non-negative integers n and k to be
$$\binom{n}{k} = \frac{(n)(n-1)...(n-k+1)}{(k)(k-1)...(1)} = \frac{n!}{k!(n-k)!}$$
To find the q-analog of this, we replace each factor r by
$$\frac{1-q^r}{1-q},$$
which leads us to the following definition.

Definition 2.1. The q-binomial (or Gaussian) coefficient is defined for non-negative integers n and k, $n > k$, as
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q[n-1]_q....[n-k+1]_q}{[k]_q[k-1]_q...[1]_q} = \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})...(1-q)}$$

Notice that division is exact here, so this is a polynomial and not a rational function. We show a few examples of computing q-binomial coefficients.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q = 1,$$
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = \frac{1-q^2}{1-q} = 1+q,$$
$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = \frac{(1-q^3)(1-q^2)}{(1-q^2)(1-q)} = 1+q+q^2,$$
$$\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = \frac{(1-q^6)(1-q^5)(1-q^4)}{(1-q^3)(1-q^2)(1-q)} = 1+q+2q^2+3q^3+3q^4+3q^5+3q^6+2q^7+q^8+q^9.$$

Proposition 2.2. *Reflection*

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{bmatrix} m+n \\ n \end{bmatrix}_q$$

This can be proved by directly using the formula.

Proposition 2.3. *Analog of Pascal's identity*

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = q^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q + \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Proof.

$$\begin{aligned} q^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q + \begin{bmatrix} n \\ k \end{bmatrix}_q &= q^{k+1} \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k})}{(1-q^{k+1})(1-q^k)...(1-q)} + \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})...(1-q)} \\ &= \left(\frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})...(1-q)}\right)\left(\frac{q^{k+1}(1-q^{n-k})}{1-q^{k+1}}+1\right) \\ &= \left(\frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})...(1-q)}\right)\left(\frac{(q^{k+1}-q^{n+1})+1-q^{k+1}}{1-q^{k+1}}\right) \\ &= \left(\frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})...(1-q)}\right)\left(\frac{1-q^{n+1}}{1-q^{k+1}}\right) \\ &= \left(\frac{(1-q^{n+1})(1-q^n)...(1-q^{n-k+1})}{(1-q^{k+1})(1-q^k)...(1-q)}\right) \\ &= \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \end{aligned}$$

■

Proposition 2.4. *Now we will see another interpretation of q-binomial coefficients, involving Ferrers boards. Let $p_k(m,n)$ denote the number of Ferrers boards with k squares on a complete $m \times n$ chess board. Consider the generating function,$P(m,n)$, for the number of Ferrers boards with k squares on a complete $m \times n$ chess board.*

$$P(m,n) = \sum_{i=0}^{\infty} p_i(m,n)q^i = \begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{bmatrix} m+n \\ n \end{bmatrix}_q$$

Note that this sum is actually finite, as for $i > mn$, $p_i(m,n) = 0$

Proof. If we show that

$$P(0,0) = 0 \text{ and } P(0,1) = 1 \text{ and } P(m,n) = q^n P(m-1,n) + P(m,n-1)$$

then we have proved that $P(m,n) = \begin{bmatrix} m+n \\ m \end{bmatrix}_q$ as $P(m,n)$ follows the same recurrence as $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ and $P(m,n)$ also has the same starting values.

Number of Ferrers board with 0 squares in a 0×0 chess board is 1, and for all other positive integers k, the number of Ferrers boards with k squares in a 0×0 chess board is 0. Number of Ferrers boards with 0 squares in a 1×0 chess board is 1, and for all other positive integers k, the number of Ferrers boards with k squares in a 1×0 chess board is 0.

So, $P(0,0) = P(1,0) = 1$. Now, we must only prove the recurrence relation to complete our proof.

Observe that each lattice path from $(0,0)$ to (m,n) where we can move only in the north direction or in the east direction determines a Ferrers board - The board under the path. Using this interpretation of Ferrers board, we will prove the recurrence relation. To prove the recurrence relation, we need to show that the coefficients of q^i on both sides are equal. So, we need to prove $p_i(m,n) = p_{i-n}(m-1,n) + p_i(m,n-1)$.

The Ferrers board with i squares on a $m \times n$ board either contains the bottom right square or it doesn't. Ferrers boards not containing the bottom right square are contained in a $m \times (n-1)$ rectangle.So, number of such Ferrers boards is $p_i(m,n-1)$. Suppose the Ferrers board contains the bottom right corner square. Then, the entire bottom row is included in the Ferrers board. Hence, the number of such Ferrers boards is equivalent to the number of Ferrers boards(with $i-n$ squares) excluding the bottom row of the $m \times n$ square. Hence there are $p_{i-n}(m-1,n)$ such Ferrers boards. This completes the proof of the recurrence relation. ■

Theorem 2.5. *Analog of Binomial theorem*
The q-binomial theorem states that if x and q are some non-negative integers, then

$$\prod_{i=0}^{n-1}(1+xq^i) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k$$

Proof. We will prove the finite q-binomial theorem, using the proof given by G. Polya and G.L. Alexanderson [2]. Denote the LHS by $f(x)$. We will write

$$f(x) = (1+x)(1+xq)....(1+xq^{n-1}) = \sum_{k=0}^{n} Q_k x^k \tag{2.1}$$

Note that, $Q_0 = 1$ and $Q_n = q^{n(n-1)/2}$
Further, (2.1) implies

$$(1+x)f(qx) = (1+q^n x)f(x)$$

Substituting the value of $f(x)$ in the above equation, we get

$$(1+x)\sum_{k=0}^{n} Q_k q^k x^k = (1+q^n x)\sum_{k=0}^{n} Q_k x^k$$

Comparing coefficients of x^r, we get

$$Q_r q^r + Q_{r-1} q^{r-1} = Q_r + q^n Q_{r-1},$$

or

$$Q_r = Q_{r-1}\frac{q^{n-r+1}-1}{q^r-1} q^{r-1}. \tag{2.2}$$

Since $Q_0 = 1$ and $Q_n = q^{n(n-1)/2}$, we can repeatedly apply (2.2) and conclude that

$$Q_k = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2}.$$

Therefore we have

$$\prod_{i=0}^{n-1}(1 - xq^i) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k,$$

as desired. ■

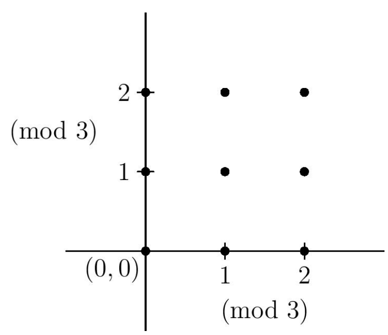
3. Counting Lines in Finite Geometries

We now shift our focus to an application of q-analogues to counting the number of lines in finite geometries; we begin by deriving an analogy of the game SET from finite fields, and proceed to show that the number of lines in a finite geometry can be counted by a q-analogue [3].

The card game SET consists of 81 cards, each of which has four attributes: color, shape, shading, and number of shapes. We can think of these attributes as the x, y, z and t axes in four dimensional space; since each of these attributes can take on three values (shading: none, striped, solid; color: red, green purple; shape: oval, diamond, squiggle; number of shapes: one, two, three), we can think of them as elements of the finite field $\mathbb{F}_3$. Similarly, we can think of a SET card as a point in $\mathbb{F}_3^4$ (a $3 \times 3 \times 3 \times 3$ grid of points).

Definition 3.1 (Finite field)**.** A *finite field* is defined as a field that contains a finite number of elements; the operations multiplication, addition, subtraction, and division are defined on such a field, as with all types of fields.

In this game, a *set* is a certain combination of three cards in which either 0, 1, 2, or 3 (but not 4) attributes are the same across the three cards. In our analogue of the game of SET to $\mathbb{F}_3^4$, a set is analogous to a line in $\mathbb{F}_3^4$. One application of q-analogues is counting the number of lines in a finite geometry such as this. Since we are working in the finite field $\mathbb{F}_3^4$, all points will be considered modulo 3. For instance, consider the finite field $\mathbb{F}_3^2$ depicted below:



The line $x + y = 0$ has solutions $(-1, 1), (0, 0), (2, -2) \ldots$. Because we are working in mod 3, we have the following mapping: $(0, 0) \to (0, 0), (-1, 1) \to (2, 1)$, and $(2, -2) \to (2, 1)$. Therefore the line $x + y = 0$ has the solutions $(0, 0), (1, 2)$, and $(2, 1)$ in $\mathbb{F}_3^2$.

Proposition 3.2. *The number of lines in the finite field $\mathbb{F}_3^2$ is 12.*

Proof. We begin by counting the number of lines passing through $(0, 0)$; since a line is defined by 2 unique points, there are $3^2 - 1 = 8$ other points to choose from. We must now divide by two, since the order of choosing the points is irrelevant. There are thus 4 lines passing through $(0, 0)$. Through computation, we find that for each equation of the form $ax + by = 0$, there are 2 equations of the form $ax + by = 1$ not passing through $(0, 0)$ and similarly 2 equations of the form $ax + by = 2$ not passing through $(0,0)$. There are thus 12 lines in total, as desired. ∎

Following the same logic, we come across the following proposition:

Proposition 3.3. *The number of lines in the finite field* $\mathbb{F}_q^n$ *is* $q\frac{q^n-1}{q-1}$.

Proof. We use the same argument as before. We have $q^n - 1$ ways to choose a point besides $(0,0)$. Then, we must divide by $q-1$ since the order of choosing the points does not matter. Finally, we add a factor of q to account for the lines $ax+by = 0, ax+by = 1, \dots ax+by = q$. This gives a total of

$$q\frac{q^n - 1}{q - 1}$$

lines in the finite field $\mathbb{F}_q^n$, as claimed. Note that this is a q-analogue of n by definition, since we have

$$\lim_{q\to 1} q\frac{q^n - 1}{q - 1} = n.$$

∎

4. q-Calculus

One of the most prominent applications of q-analogues is in q-calculus, or quantum calculus. In this section, we provide a brief exposition on some of the foundations of q-calculus. We begin with the differential and the derivative.

Definition 4.1. We define the q-*derivative* of a function $f(x)$ as

$$d_q f(x) = f(qx) - f(x).$$

Definition 4.2. With this definition of the q-derivative, we can now define the q-*differential* D_q [4] as follows:

$$D_q(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{qx - x}.$$

Note that when we take the limit as $q \to 1$ of the q-differential, we arrive at the definition of a derivative in classical calculus. Similarly, let us now define the q-analogue to the classical antiderivative.

Definition 4.3. We define the q-antiderivative [5], $F(x)$, of a function $f(x)$ such that $D_q F(x) = f(x)$. We write

$$F(x) = \int f(x) d_q x.$$

As in the case of classical calculus, this q-antiderivative is not unique: adding a function of the type $g(x)$ where $g(qx) = g(x)$ will not change the derivative since $D_q g(x) = 0$ for such a $g(x)$. We now state an important proposition regarding the uniqueness of the q-antiderivative.

Proposition 4.4. *Let* $q \in (0,1)$. *Then up to adding a constant, a function* $f(x)$ *has no more than one* q*-antiderivative that is continuous at* $x = 0$.

Proof. We prove this idea using contradiction and methods from real analysis [5]. Suppose, for the sake of contradiction, that $F_1(x), F_2(x)$ are two q-antiderivatives of $f(x)$ (which are continuous at $x = 0$). Let $g(x) = F_1(x) - F_2(x)$; since $D_q g(x) = 0$, we have $g(qx) = g(x)$. Now for an arbitrary $K > 0$, define m and M as follows:

$$m = \inf\{g(x) \mid qK \le x \le K\},$$
$$M = \sup\{g(x) \mid qK \le x \le K\}.$$

We can assume that $m < M$, and so that either $g(0) \neq m, g(0) \neq M$, or both. Without loss of generality, let us assume that $g(0) \neq m$; then since g is continuous at $x = 0$, there will always be a $\delta > 0$ for sufficiently small ε for which

$$m + \varepsilon \notin g(0, \delta). \tag{1}$$

However, for sufficiently large N, we have $q^N \cdot K < \delta$. We additionally utilize the fact that $g(qx) = g(x)$ to find that

$$m + \varepsilon \in (m, M) \subset g[qK, K] = g[q^{N+1}K, q^N K] \subset g(0, \delta).$$

This is clearly a contradiction to (1); therefore, $m = M$ and thus $g(x)$ is constant in $[qK, K]$ and subsequently over $\mathbb{R}$. ■

We now have a definition for the q-antiderivative, and we need a method to compute it. For this, we turn to the fundamental theorem of q-calculus, an analogue of the fundamental theorem of classical calculus.

Theorem 4.5 (Fundamental theorem of q-Calculus)**.** *Given that $F(x)$, the q-antiderivative of $f(x)$ is continuous at $x = 0$, we have*

$$\int_a^b f(x) d_q x = F(b) - F(a)$$

for $a < b$ and $a, b \in [0, \infty)$.

Proof. $F(x)$ is given by the Jackson formula[1], meaning

$$\begin{aligned}(1-q)a\sum_{j=0}^{\infty} q^j f(q^j a) &= F(x) - F(0)\\ &= \int_0^a f(x) d_q x.\end{aligned}$$

Thus by definition of the Jackson integral, we have

$$\int_0^a f(x) d_q x = F(a) - F(0). \tag{1}$$

And similarly,

$$\int_0^b f(x) d_q x = F(b) - F(0). \tag{2}$$

Subtracting (1) from (2), we find that

$$\int_a^b f(x) d_q x = F(b) - F(a)$$

[1]Visit this link for a basic overview of the Jackson integral.

as desired. ■

Lastly, we will look at the q-analogue to the method of integration by parts in classical calculus. Let f and g be two differentiable functions that are continuous at $x = 0$. The derivative of their product is then given by

$$D_q\left(f(x)g(x)\right) = f(x)\left(D_q g(x)\right) + g(qx)\left(D_q f(x)\right).$$

To proceed, we will need to make use of the following corollary of Theorem 4.5: if $f'(x)$ is continuous at $x = 0$, then

$$\int_a^b D_q f(x) d_q x = f(b) - f(a).$$

Using this result, we have

$$f(b)g(b) - f(a)g(a) = \int_a^b f(x)(D_q g(x)) d_q x + \int_a^b g(qx)(D_q f(x)) d_q x.$$

Rearranging this gives

$$\int_a^b f(x) d_q g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) d_q f(x). \tag{4.1}$$

This formula can be used for q-integration by parts. This method is integral to the derivation of the q-Taylor formula with the Cauchy remainder term.

5. q-NUMBERS

5.1. q-Narayana numbers.

The q-Narayana numbers were orignally discovered by MacMahon and then were rediscovered by Narayana. Interestingly , it turns out that many statistics of combinatorial structures have the Narayan distribution.

Definition 5.1. The q-Narayana numbers are defined as

$$N(n,k;q) = \frac{1}{[k]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

Proposition 5.2.

$$N(n,k;q) = q^{k(k-1)-n} \left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \right)$$

Definition 5.3. The Young tableau of a Ferrers diagram is obtained by placing the numbers 1, ..., n in the n boxes of the diagram. A standard Young tableau is a Young tableau in which the numbers form an increasing sequence along each line and along each column. A Young tableau in which numbers are non-decreasing along lines and increasing along columns is called a *semistandard Young tableau* (SSYT).The index $\boldsymbol{l}$ is called the length, $\boldsymbol{l}(\lambda)$ of λ. A semistandard Young tablue of shape λ is an array $T = (T_{ij})$ of positive, where $1 \leqslant i \leqslant \boldsymbol{l}(\lambda)$ and $1 \leqslant j \leqslant \lambda_i$, that is weakly increasing in rows and strictly increasing in columns.

Theorem 5.4. *For any $n > 0$ and $S \subseteq [2n-1], [S] = k$ we have that $\beta_n(S)$ counts the number of SSYT's of shape $\langle 2^k \rangle$ with row$(T) = S$ and with parts less than n.*

Proof. The proof of the theorem is omitted, but can be found in the following link: link ■

Proposition 5.5.

$$N(n, k+1; q) = S_{2^k}(q, q^2, q^3, \ldots q^{n-1})$$

Proof. By Theorem 4.3 we have that

$$\sum_{w \in D_n} des(w) = kq^{MAJ(w)} = \sum_{|S|=k} \beta_n(S) q^{\sum_{s \in S} s} = \sum T q^{\sum T_{ij}},$$

where the last sum is over all SSYT's of shape $\langle 2^k \rangle$ with parts less than n. By the combinatorial definition of the Schur function this is equal to $N(n, k+1; q) = S_{2^k}(q, q^2, q^3, \ldots q^{n-1})$ and the theorem follows. ∎

Definition 5.6. q - Catalan numbers are defined by

$$C_n(q) = \sum_{k=0}^{n-1} k_{C_k}(q) C_{n-k-1}(q)$$

with $C_0(q) = 1$ Let

$$f(z, q) = \sum_{k \geqslant 0} C_k(q) z^k$$

be their generating function, which can uniquely be determined by the functional equation

$$f(z, q) = 1 + z f(z, q) f(qz, q).$$

This implies a well known fact that it can be represented in the form

$$f(z, q) = \frac{E_2(-qz)}{E_2(-z)}$$

Here $E_r(z)$ denotes the generalized q - Exponential function

$$E_r(z) = \sum_{k \geqslant 0} q^{r\binom{k}{2}} \frac{z^k}{(1-q)(1-q^2)\ldots(1-q^k)}$$

$$E_r(z) - E_r(qz) = z E_r(q^r z)$$

For $n \in N$ we define

$$G_r(z, n) = \sum_{k \geqslant 0} G(k, n, r) z^k := \frac{E_r(-q^n z)}{E_r(-z)}$$

Then we have

$$G_r(z, n+1) = G_r(z, n) + q^n z G_r(z, n+r)$$

We compare coefficients we get

$$\frac{G(k, n+1, r) - G(k, n, r)}{q^n} = G(k-1, n+r, r)$$

with $G(k, 0, r) = [k = 0]$ and $G(0, n, r) = 1$ This implies

$$G_r(z, 1) = 1 + z G_r(z, r)$$

These are the characteristics properties of the q-Gould polynomial. For $q = 1$ they have the explicit formula $G(k, n, r) = \frac{n}{n+rk}\binom{n+rk}{k}$.

For general q special values are $G(k,n,0) = q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}$ and $G(k,n,1) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}$ where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes a q - Binomial coefficient. For $r > 1$ no explicit formulas are known. Also note that $G_2(z,1) = 1 + zG_2(z,2)G_2(qz,1) = f(z,q)$ is the generating function of q - Catalan numbers. From

$$\frac{E_r(-q^n z)}{E_r(-z)} = \frac{E_r(-qz)}{E_r(-z)}\frac{E_r(-q^2 z)}{E_r(-qz)} \cdots \frac{E_r(-q^n z)}{E_r(-q^{n-1}z)}$$

we get

$$G_r(z,n) = G_r(z,1)G_r(qz,1)...G_r(q^m z,n)$$

and

$$G_r(z,m+n) = G_r(z,m)G_r(q^m z,n)$$

5.2. q-Lucas number.

Theorem 5.7. *Here we will be defining q-Lucas theorem. Let*

$$n = n_1 d + n_0$$

and

$$k = k_1 d + k_0$$

where

$$0 \le n_0, k_0 < d$$

Then we get that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv \binom{n_1}{k_1}\begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_q \pmod{\Phi_d}$$

Here Φ_d is the d^{th} cyclotomic polynomial [2].
Then we get that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv \binom{n_1}{k_1}\begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_q$$

This is only when $q = e^{\frac{2c\pi i}{d}}$. With $\gcd(c,d) = 1$.

Proof. Let $S = \binom{n}{k}$. Now if $\omega \in \binom{n}{k}$, then we can write it as $\omega = \omega_1\omega_2$. Here ω_1 denotes the first d bits of ω and ω_2 is the remaining suffix. Our goal is to show that S^G is given by right side of the recurrence. Let us pick a random $\omega \in S^G$. Then ω_1 is either all 0's or all 1's. In the first case , $\mathrm{inv}\omega = \mathrm{inv}\omega_2$ In the second case $\mathrm{inv}\omega = \mathrm{inv}\omega_2 + kd$. Thus we always have

$$q^{\mathrm{inv}\omega} \equiv q^{\mathrm{inv}\omega_2}$$

As we are working $\pmod{\Phi_d}$, this sets q to be a primitive d^{th} root of unity. Moving ahead

$$\omega_2 \in \widehat{\binom{n-k}{k-d}}.$$

This is in the first case and

$$\omega_2 \in \widehat{\binom{n-k}{k}}$$

in the second case. Hence $[S^G]$ does give the right generating function. Finally we show that any other orbit $\mathscr{O}$ has weight divisible by Φ_d. Take $\omega \in \mathscr{O}$ and let us assume ω_1 contains l zeros. Then for the generator g of C_d we have

$$\text{inv} g\omega \equiv \text{inv}\omega + l \pmod{d}. \tag{1}$$

As the number of inversions either goes up by l or goes down by $d-l$. Now if $m = \#\mathscr{O}$, then

$$[\mathscr{O}] = q^{\text{inv}\omega}(q^l + q^{2l} + ... + q^{ml}). \tag{2}$$

But we know that $\omega = g^m\omega$ forces $d|ml$ by repeated application of (1). Thus the right side of (2) is divisible by Φ_d. ■

5.3. q-Pochhammer symbol.

Definition 5.8. The q-Pochhammer symbol is defined as

$$(a;q)_n := (1-a)(1-aq)...(1-aq^{n-1})$$

where $(a;q)_0 := 1$

Another name for it is q - shifted factorial

we get the relation

$$(q^a;q)_n + r = (q^a;q)_r(q^{a+r};q)_n$$

By convention

$$(a;q)_\infty := \Pi_{j\geqslant 0}(1-aq^j)$$

so that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

This lets us set

$$(a;q)_{-n} := \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}.$$

On can also prove that,

$$(aq^{-n};q)_n = q^{-\binom{n}{2}}(\frac{-a}{q})^n(\frac{q}{a};q)_n$$

Some other identities are

$$[m]_q = \frac{(q;q)_m}{(1-q)(q;q)_{m-1}} = \frac{(q^m;q)_\infty}{(1-q)(q^{m+1};q)_\infty}$$

$$[m]_q! = \frac{(q;q)_m}{(1-q)^n}$$

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{(q^{m-r+1};q)_r}{(q;q)_r} = \frac{(q^{r+1};q)_\infty(q^{m-r+1};q)_\infty}{(q^{m+1};q)_\infty(q;q)_\infty}$$

5.4. **q-Bernoulli polynomials.** Bernoulli polynomials are of significant importance in mathematics and physics. The reason is that Bernoulli polynomials arise in many applications.q-Bernoulli polynomials posses many interesting properties which can be used in many areas.

Definition 5.9. In this unit we will learn more on q-Bernoulli polynomials.

$$[x]_q = \frac{1-q^x}{1-q}$$

for any real number x. Let us assume that $q \in \mathrm{C}$ with $|q| < 1$. Let

$$F_{q^r}(t) = \frac{q^r - 1}{r \log q} e^{\frac{t}{1-q^r}} - t \sum_{n=0}^{\infty} q^{rn} e^{[n]_{q^r} t}, |t| < 1$$

Consider the Taylor expansion at $t = 0$.

$$F_{q^r}(t) = \beta_{0,q^r} + \beta_{1,q^r} \frac{t}{1!} + \beta_{2,q^r} \frac{t^2}{2!} + ... + \beta_{n,q^r} \frac{t^n}{n!} + ...$$

The coefficients β_{n,q^r} are called n^{th} q-Bernoulli numbers.

6. Hypergeometric Series

6.1. **Introduction.** Our main objective in this section is to present the definitions and notations for hypergeometric and basic hypergeometric series, and to derive the elementary formulas that form the basis for most of the summation, transformation and expansion formulas and basic integrals. We begin by defining Gauss ${}_2F_1$ hypergeometric series, the ${}_rF_s$ (generalized) hypergeometric series, and pointing out some of their important special cases.

Next, we define the Heine's ${}_2\phi_1$ basic hypergeometric series which contains an additional parameter q, also called the base, and then give the definition and notations for ${}_r\phi_s$ basic hypergeometric series. Basic hypergeometric series are called q-analogues of hypergeometric series because an ${}_rF_s$ series can be obtained as the $q \to 1$ limit case of an ${}_r\phi_s$ series.

Next, we use the q-binomial theorem (stated in the previous section), to derive Heine's q-analogues of Euler's transformation formulas, Jacobi's triple product identity, and summation formulas that are q-analogues of those for hypergeometric series due to Chu and Vandermonde, Gauss, Kummer, Plaff and Saalschutz, and to Karlsson and Minton.

We also introduce q-analogues of the exponential, gamma and beta functions, as well as the concept of a q-integral that surprisingly allows us to give a q-analogue of the famous Euler's integral representation of a hypergeometric function.

1.2. **Hypergeometric and Basic Hypergeometric Series** In 1812, Gauss presented to the Royal Society of Sciences at Gottingen his famous paper [6] in which he considered the infinite series

$$1 + \frac{ab}{1 \cdot c} z + \frac{a\,(a+1)\,b\,(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a\,(a+1)\,(a+2)\,b\,(b+1)\,(b+1)}{1 \cdot 2 \cdot c\,(c+1)\,(c+2)} z^3 + \cdots \tag{6.1}$$

as a function of a, b, c and z where it is assumed that $c \notin \mathbb{Z} \setminus \{\mathbb{N}\}$ and $c \neq 0$ so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for $|z| < 1$. Although Gauss used the notation $F(a, b, c, z)$ for his series, it is now customary to use $F(a,b;c;z)$ or ${}_2F_1(a,b;c;z)$ for this series (and for its sum when it converges), because these notations separate the numerator parameters a,b from the denominator parameter c and the variable z. In view of Gauss paper, his series is frequently called *Gauss series*. However, since the special case $a=1, b=c$ yields the geometric series

$$1+z+z^2+z^3+z^4+\cdots$$

Gauss series is also called the *(ordinary) hypergeometric series* or the *Gauss hyper geometric series*. Some important functions which can be expressed by means of Gauss' series are

$$(1+z)^a = F(-a,b;b;-z), \quad \log(1+z) = zF(1,1;2;-z),$$

$$\arcsin(z) = zF\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right), \quad \arctan(z) = zF\left(\frac{1}{2},1;\frac{3}{2};-z^2\right), \quad e^z = \lim_{a\to\infty} F(a,b;b;z/a).$$

where $|z|<1$ in the first four formulas. Also expressible by means of Gauss' series are the classical orthogonal polynomials, such as the *Tchebichef polynomials* of the first/second kind

$$T_n(x) = F\left(-n,n;\frac{1}{2};\frac{(1-x)}{2}\right),$$

$$U_n(x) = (n+1)F\left(-n,n+2;\frac{3}{2};\frac{(1-x)}{2}\right),$$

The *Legendre polynomials*

$$P_n(x) = F\left(-n,n+1;1;\frac{(1-x)}{2}\right),$$

the *Gegenbauer (ultraspherical) polynomials*

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{n!}F\left(-n,n+2\lambda;\lambda+\frac{1}{2};\frac{(1-x)}{2}\right)$$

and the more general *Jacobi polynomials*

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}F\left(-n,n+\alpha+\beta+1;\alpha+1;\frac{(1-x)}{2}\right)$$

where $n\in\mathbb{N}$ and $(\alpha)_n$ denotes the *shifted factorial* defined by

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

for all $n\geqslant 1$. Before Gauss, Chu [1303] (also see [7], [8] and [10, Page 59]) and [9] had proved the summation formula

$$F(-n,b;c;1) = \frac{(c-b)_n}{(c)_n} \tag{6.2}$$

which is now called *Vandermonde's formula* or the *Chu-Vandermonde formula*, and Euler had derived several results for hypergeometric series, including his transformation formula

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z), \;\; |z|<1. \tag{6.3}$$

Notice that, equation (6.2) is a terminating case $a=-n$ of the summation formula :

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

which Gauss proved in his paper. 33 years later, Heine [11], [12] and [13] introduced the series

$$1+\frac{\left(1-q^{a}\right)\left(1-q^{b}\right)}{(1-q)\left(1-q^{c}\right)} z+\frac{\left(1-q^{a}\right)\left(1-q^{a+1}\right)\left(1-q^{b}\right)\left(1-q^{b+1}\right)}{(1-q)\left(1-q^{2}\right)\left(1-q^{c}\right)\left(1-q^{c+1}\right)} z^{2}+\cdots \tag{6.4}$$

where as before $c \notin \mathbb{Z} \setminus \{\mathbb{N}\}$ and $q \neq 1$. This series converges absolutely for $|z|<1$ when $|q|<1$ and it tends (at least termwise) to Gauss series as $q \mapsto 1$, simply because

$$\lim_{q \rightarrow 1} \frac{\left(1-q^{\alpha}\right)}{(1-q)}=\alpha$$

The series is usually called Heine's *series* or in view of the base q, *q-hypergeometric series*. Analogous to Gauss notation, Heine used the notation $\phi(a, b, c, q, z)$ for his series. However, since one would like to also be able to consider the case when q to the power a, b or c is replaced by zero, it is now customary to define the basic hypergeometric series by

$$\phi(a, b ; c ; q, z) \equiv {}_{2}\phi_{1}(a, b ; c ; q, z) \equiv {}_{2}\phi_{1}\left[\begin{array}{c} a, b \\ c \end{array} ; q, z\right] \equiv \sum_{n=0}^{+\infty} \frac{(a ; q)_{n}(b ; q)_{n}}{(q ; q)_{n}(c ; q)_{n}} \tag{6.5}$$

where $(a;q)_n$ represents the q-Pochammer symbol (also known as the *q-shifted factorial*). Note that, in equation (6.5) it is assumed that $c \neq q^{-m}$ where $m \in \mathbb{N}$.

Another generalization of Gauss series is the (*generalized*) *hypergeometric series* with r numerator parameters $a_1, a_2, \ldots, a_r$ and s denominator parameters $b_1, b_2, \ldots, b_s$ defined by

$${}_{r}F_{s}(a_1, a_2, \ldots, a_r ; b_1, b_2, \ldots b_s ; z) \equiv {}_{r}F_{s}\left[\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array} ; z\right]=\sum_{n=1}^{+\infty} \frac{(a_1)_n (a_2)_n \ldots (a_r)_n}{n! (b_1)_n (b_2)_n \ldots (b_s)_n} z^n$$

Some well-known special cases are the trigonometric functions

$$\sin(z)=z\,{}_{0}F_{1}(-;3/2;-z^{2}/4), \quad \cos(z)=z\,{}_{0}F_{1}(-;1/2;-z^{2}/4),$$

including the famous *Bessel function*

$$J_{\alpha}(z)=\left(\frac{z}{2}\right)^{\alpha} \frac{{}_{0}F_{1}(-;\alpha+1;-z^{2}/4)}{\Gamma(\alpha+1)}$$

where a dash is used to indicate the absence of either numerator (when $r=0$) or denominator (when $s=0$) parameters. Some other well-known special cases are the *Hermite polynomials*

$$H_{n}(x)=(2x)^{n}{}_{2}F_{1}\left(-\frac{n}{2}, \frac{(1-n)}{2} ;-;-\frac{1}{x^{2}}\right)$$

and the *Laguerre polynomials*:

$$L_{n}^{\alpha}(x)=\frac{(\alpha+1)_{n}}{n!}\,{}_{1}F_{1}(-n;\alpha+1;x).$$

Generalizing Heine's series, we shall define an ${}_r\phi_s$ *basic hyper geometric series* by

$${}_{r}\phi_{s}(a_1, a_2, \ldots, a_r ; b_1, b_2, \ldots b_s ; q, z)=\sum_{n=1}^{+\infty} \frac{(a_1 ; q)_n (a_2 ; q)_n \ldots (a_r ; q)_n}{n! (b_1 ; q)_n (b_2 ; q)_n \ldots (b_s ; q)_n}\left[(-1)^{n} q^{\binom{n}{2}}\right]^{s-r+1} z^{n}$$

Notice that, we have assumed that the parameters $b_1, b_2, \ldots, b_s$ are such that the denominator factors in the terms of the series are never zero. Since

$$(-m)_n = (q^{-m}; q)_n = 0, \quad n = m+1, m+2, m+3, \ldots,$$

an ${}_rF_s$ series terminates if one of its numerator parameter is zero or a negative integer, and an ${}_r\phi_s$ series terminates if one of its numerator parameter is of the form q^{-m} with $m \in \mathbb{N}$.

For all $n \in \{\mathbb{N} \cup 0\}$, we also define

$$(a; q)_\infty = \prod_{k=0}^{+\infty} \left(1 - aq^k\right), \ |q| < 1$$

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notations

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \ldots (a_m; q)_n$$

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \ldots (a_m; q)_\infty.$$

The ratio $(1 - q^a)/(1 - q)$ is called a *q-number* (or *basic number*) and it is denoted by

$$[a]_q = \frac{1 - q^a}{1 - q}, \ \ q \neq 1.$$

It is also called a q-analogue, q-deformation, q-extension, or a q-generalization of the complex number a. In terms of q-numbers the q-number factorial $[n]_q!$ is defined for a $n \in \mathbb{N}$ by

$$[n]_q! = \prod_{k=1}^{n} [k]_q$$

and the corresponding q-number shifted factorial is defined by

$$[a]_{q;n} = \prod_{k=0}^{n-1} [a + k]_q$$

It's not hard to see that

$$\lim_{q \to 1} [n]_q! = n!, \quad \lim_{q \to 1} [a]_q = a, \quad \lim_{q \to 1} [a]_{q;n} = (a)_n.$$

We can use the compact notation

$$[a_1, a_2, \ldots, a_m]_{q;n} = [a_1]_{q;n} [a_2]_{q;n} \ldots [a_m]_{q;n}$$

Therefore we find that

$$\sum_{n=0}^{+\infty} \frac{[a_1, a_2, \ldots, a_r]_{q;n}}{[n]_q! [b_1, b_2, \ldots, b_s]_{q;n}} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n = {}_r\phi_s(q^{a_1}, \ldots, q^{a_r}; q^{b_1}, \ldots, q^{b_s}; q, z(1-q)^{1+s-r})$$

As in Frenkel and Turaev [14] one can define a *trigonometric number* $[a; \sigma]$ by

$$[a; \sigma] = \frac{\sin(\pi \sigma a)}{\sin(\pi \sigma)}$$

for some non-integer values of σ and view $[a;\sigma]$ as a trigonometric deformation of a since $\lim_{\sigma\to 0}[a;\sigma] = a$. The corresponding ${}_rt_s$ *trigonometric hyper geometric series* is defined by

$$ {}_rt_s(q^{a_1},\ldots,q^{a_r};q^{b_1},\ldots,q^{b_s};\sigma,z) = \sum_{n=0}^{+\infty} \frac{[a_1,a_2,\ldots,a_r]_n}{[n;\sigma]![b_1,b_2,\ldots,b_s]_n} \left[(-1)^n e^{\pi i\sigma\binom{n}{2}}\right]^{1+s-r} z^n $$

where

$$ [n;\sigma]! = \prod_{k=0}^{n}[k;\sigma], \quad [a;\sigma]_n = \prod_{k=0}^{n-1}[a+k;\sigma] $$

and

$$ [a_1,a_2,\ldots,a_m;\sigma]_n = [a_1;\sigma]_n[a_2;\sigma]_n\ldots[a_m;\sigma]_n $$

From the fact that

$$ [a;\sigma] = \frac{e^{\pi i\sigma a}-e^{-\pi i\sigma a}}{e^{\pi i\sigma}-e^{-\pi i\sigma}} = \frac{q^{a/2}-q^{-a/2}}{q^{1/2}-q^{-1/2}} = \frac{1-q^a}{1-q}\, q^{(1-a)/2} $$

where $q \mapsto e^{2\pi i\sigma}$, it follows that

$$ [a;\sigma]_n = \frac{(q^a;q)_n}{(1-q)^n}\, q^{n(1-a)/2-n(n-1)/4} $$

and hence we deduce that

$$ {}_rt_s(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots b_s;\sigma,z) = {}_r\phi_s(q^{a_1},q^{a_2},\ldots,q^{a_r};q^{b_1},q^{b_2},\ldots,q^{b_s};q,cz) $$

where $c = (1-q)^{1+s-r}q^{r/2-s/2+(b_1+b_2+\ldots+b_s)/2-(a_1+a_2+\ldots+a_r)/2}$.

6.2. **Heine's transformation formulas for ${}_2\phi_1$ series.** From the last section, we know that q-Binomial theorem states that

$$ {}_1\phi_0(a;-;q,z) = \sum_{n=0}^{+\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1, |q|<1. \tag{6.6} $$

Heine [11], [12] and [13] showed that

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\, {}_2\phi_1(c/b,z;az;q,b), \quad |z|<1, |b|<1. \tag{6.7} $$

To prove this transformation formula, first observe from the q-binomial theorem (6.6) that

$$ \frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} = \sum_{m=0}^{+\infty} \frac{(c/b;q)_m}{(q;q)_m} (bq^n)^m. $$

Hence for $|z| < 1$ and $|b| < 1$ we deduce that

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{+\infty}\frac{(a;q)_n(cq^n;q)_\infty}{(q;q)_n(bq^n;q)_\infty}\,z^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{+\infty}\frac{(a;q)_n}{(q;q)_n}\,z^n\sum_{n=0}^{+\infty}\frac{(cq^n;q)_\infty}{(bq^n;q)_\infty}\,z^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{+\infty}\frac{(a;q)_n}{(q;q)_n}\,z^n\sum_{m=0}^{+\infty}\frac{(c/b;q)_m}{(q;q)_m}\,(bq^n)^m \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{+\infty}\frac{(c/b;q)_m}{(q;q)_m}\,b^m\sum_{n=0}^{+\infty}\frac{(a;q)_n}{(q;q)_n}\,(zq^m)^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{+\infty}\frac{(c/b;q)_m}{(q;q)_m}\,b^m\,\frac{(azq^m;q)_\infty}{(zq^m;q)_\infty}
\end{aligned}
$$

Now it's not hard to see that

$$
\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{+\infty}\frac{(c/b;q)_m}{(q;q)_m}\,b^m\,\frac{(azq^m;q)_\infty}{(zq^m;q)_\infty} = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1(c/b,z;az;q,b)
$$

as desired. This completes the proof of Heine's transformation formulas for ${}_2\phi_1$ series.

Heine also showed that Euler's transformation formula which states that

$$
{}_2F_1(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z), \quad |z| < 1.
$$

has a q-analogue of the form

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1(c/b,c/b;c;q,abc/z). \tag{6.8}
$$

A short and quite interesting way to prove this beautiful formula is just to iterate the Heine's transformation formulas for ${}_2\phi_1$ series (see equation (6.7)) as follows:

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1(c/b,z;az;q,b) \\
= \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1(abz/c,b;bz;q,c/b) &= \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1(c/a,c/b;c;q,abz/c)
\end{aligned}
$$

Therefore we deduce that

$$
{}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1(c/b,c/b;c;q,abc/z)
$$

as desired. This completes the proof of q-analogue of Euler's transformation formula.

6.3. **Heine's q-analogue of Gauss' summation formula.** In order to derive Heine's [11], [12] q-analogue of Gauss summation formula it suffices to set $z \mapsto c/ab$ in Heine's transformation formulas for ${}_2\phi_1$ series (6.7) assuming that $|b| < 1$, $|c/ab| < 1$and simply observe that the series on the right hand side of

$$
{}_2\phi_1(a,b;c;q,c/ab) = \frac{(b,c/b;q)_\infty}{(c,c/ab;q)_\infty}\,{}_1\phi_0(c/ab;-;q,b)
$$

can be summed by using the q-binomial theorem to give

$$ {}_2\phi_1(a,b;c;q,c/ab) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty} \tag{6.9} $$

Notice that, by analytic continuation, we may drop the assumption that $|b| < 1$ and require only that $|c/ab| < 1|$ for the previous equation (6.9) to be valid.

For the terminating case when $a = q^{-n}$, equation (80) reduces to

$$ {}_2\phi_1(q^{-n},b;c;q,cq^n/b) = \frac{(c/b;q)_n}{(c;q)_n} \tag{6.10} $$

By inversion or by changing the order of summation it follows from equation (6.9) that

$$ {}_2\phi_1(q^{-n},b;c;q,q) = \frac{(c/b;q)_n}{(c;q)_n}\, b^n \tag{6.11} $$

Both the equations (6.10) and (6.11) are the q-analogues of Vandermonde's formula (6.2). These formulas can be used to derive other important formulas.

For example, Jackson's [15] transformation formula states that

$$ \begin{aligned} {}_2\phi_1(a,b;c;q,z) &= \frac{(az;q)_\infty}{(z;q)_\infty} \sum_{k=0}^{+\infty} \frac{(a,c/b;q)_k}{(q,c;az;q)_k}\ (-bz)^k\, q^{\binom{n}{2}} \\ &= \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1(a,b;c;q,z) \end{aligned} \tag{6.12} $$

This formula is a q-analogue of the Pfaff-Kummer transformation formula:

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1(a,c-b;c;z/z(z-1)) $$

To prove Jackson's transformation formula (6.12), we use equation (6.10) to write

$$ \begin{aligned} {}_2\phi_1(a,b;c;q,z) &= \sum_{k=0}^{+\infty} \frac{(a;q)_k}{(q;q)_k}\, z^k \sum_{n=0}^{k} \frac{(q^{-k},c/b;q)_n}{(q,c;q)_n}\,(bq^k)^n \\ &= \sum_{n=0}^{+\infty}\sum_{k=n}^{+\infty} \frac{(a;q)_k(c/b;q)_n}{(q;q)_{k-n}(q,c;q)_n}\, z^k\,(-b)^n\, q^{\binom{n}{2}} \\ &= \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty} \frac{(a;q)_{k+n}(c/b;q)_n}{(q;q)_k(q,c;q)_n}\, z^k\,(-bz)^n\, q^{\binom{n}{2}} \\ &= \sum_{n=0}^{+\infty} \frac{(a,c/b;q)_n}{(q,c;q)_n}\,(-bz)^n\, q^{\binom{n}{2}} \sum_{k=0}^{+\infty} \frac{(aq^n;q)_k}{(q;q)_k}\, z^k \\ &= \frac{(az;q)_\infty}{(z;q)_\infty} \sum_{k=0}^{+\infty} \frac{(a,c/b;q)_k}{(q,c;az;q)_k}\ (-bz)^k\, q^{\binom{n}{2}} \end{aligned} $$

as desired. This completes the proof of Jackson's transformation formula.

If $a \mapsto q^{-n}$, then the series on the right side of equation (6.12) can be reversed (simply by replacing k by $(n-k)$ to yield the famous Sears transformation formula

$$ {}_2\phi_1(q^{-n}, b; c; q, z) = \frac{(c/b;q)_n}{(c;q)_n} \left(\frac{bz}{q}\right)^n {}_3\phi_2(q^{-n}, q/z; c^{-1}q^{1-n}; bc^{-1}q^{1-n}, 0; q, q) \tag{6.12}$$

6.4. **Jacobi's triple product identity and theta functions.** Jacobi's [16] well-known *triple product identity* (also see Andrews [17]) states that

$$ (z\sqrt{q}, \sqrt{q}/z, q; q)_\infty = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2/2} z^n \tag{6.13}$$

where $z \neq 0$ can be easily derived by using Heine's summation formula (6.9). Now Plugging $c \mapsto bz\sqrt{q}$ in equation (6.9) and then let $b \to \infty$ and $a \to \infty$ to finally obtain

$$\sum_{n=0}^{+\infty} \frac{(-1)^n q^{n^2/2} z^n}{(q;q)_n} = (z\sqrt{q};q)_\infty$$

Similarly substituting $c \mapsto zq$ in equation (6.9) and letting $a \to \infty$ and $b \to \infty$ produces

$$\sum_{n=0}^{+\infty} \frac{q^{n^2} z^n}{(q, zq; q)_n} = \frac{1}{(zq;q)_\infty}$$

Putting all things together produces

$$(z\sqrt{q}, \sqrt{q}/z; q)_\infty = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{(-1)^{m+n} q^{(m^2+n^2)/2} z^{m-n}}{(q;q)_m (q;q)_n}$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n q^{n^2/2} z^n}{(q;q)_n} \sum_{k=0}^{+\infty} \frac{q^{k^2} q^{nk}}{(q, q^{n+1}; q)_k} + \sum_{n=1}^{+\infty} \frac{(-1)^n q^{n^2/2} z^{-n}}{(q;q)_n} \sum_{k=0}^{+\infty} \frac{q^{k^2} q^{nk}}{(q, q^{n+1}; q)_k}$$

The Jacobi's triple product identity now follows simply by noticing that

$$\frac{1}{(q;q)_n} \sum_{k=0}^{+\infty} \frac{q^{k^2} q^{nk}}{(q, q^{n+1}; q)_k} = \frac{1}{(q;q)_n (q^{n+1};q)_\infty} = \frac{1}{(q;q)_\infty}$$

An important and interesting application of the Jacobi's triple product identity is that it can be used to express the theta functions (Whittaker and Watson [18, Chapter 21]:

$$\vartheta_1(x,q) = 2\sum_{n=0}^{+\infty} (-1)^n q^{((n+1)/2)^2} \sin(2n+1)x$$

$$\vartheta_2(x,q) = 2\sum_{n=0}^{+\infty} q^{((n+1)/2)^2} \cos(2n+1)x$$

$$\vartheta_3(x,q) = 1 + 2\sum_{n=0}^{+\infty} q^{n^2} \cos(2n)x$$

$$\vartheta_4(x,q) = 1 + 2\sum_{n=0}^{+\infty} (-1)^n q^{n^2} \cos(2n)x$$

in terms of infinite products. Simply replace $q \mapsto q^2$ in Jacobi's triple product identity and substitute $z \mapsto qe^{2ix}, -qe^{2ix}, -e^{2ix}, e^{2ix}$ respectively to finally obtain

$$\vartheta_1(x,q) = 2\sqrt[4]{q}\sin(x)\prod_{n=0}^{+\infty}(1-q^{2n})(1-2q^{2n}\cos(2x)+q^{4n})$$

$$\vartheta_2(x,q) = 2\sqrt[4]{q}\cos(x)\prod_{n=0}^{+\infty}(1-q^{2n})\left(1+2q^{2n}\cos(2x)+q^{4n}\right)$$

$$\vartheta_3(x,q) = \prod_{n=0}^{+\infty}\left(1-q^{2n}\right)\left(1+2q^{2n-1}\cos(2x)+q^{4n-2}\right)$$

$$\vartheta_4(x,q) = \prod_{n=0}^{+\infty}\left(1-q^{2n}\right)\left(1-2q^{2n-1}\cos(2x)+q^{4n-2}\right)$$

One can also think of the theta functions $\vartheta_1(x,q)$ and $\vartheta_2(x,q)$ as one-parameter deformations (generalizations) of the trigonometric functions $\sin(x)$ and $\cos(x)$, respectively.

This led Frenkel and Turaev [14] to define an *elliptic number* $[a;\sigma,\tau]$ by

$$[a;\sigma,\tau] = \frac{\vartheta_1\left(\pi\sigma a, e^{\pi i\tau}\right)}{\vartheta_1\left(\pi\sigma, e^{\pi i\tau}\right)} \tag{6.14}$$

where a is a complex number and the modular parameters σ and τ are fixed complex numbers such that $\Im(\tau) > 0$ and $\sigma \neq m + n\tau$ for integer values of m and n, so that the denominator $\vartheta_1\left(\pi\sigma, e^{\pi i\tau}\right)$ in equation (6.14) is never zero. Therefore it is clear that $[a;\sigma,\tau]$ is well-defined,

$$[-a;\sigma,\tau] = -[a;\sigma,\tau], \quad [1;\sigma,\tau] = 1$$

$$\lim_{\tau\to\infty}[a;\sigma,\tau] = \frac{\sin\left(\pi\sigma a\right)}{\sin\left(\pi\sigma\right)} = [a;\sigma] \tag{6.15}$$

Hence, the elliptic number $[a;\sigma,\tau]$ is a one-parameter deformation of the trigonometric number $[a;\sigma]$ and a two-parameter deformation of the number a. Notice that $[a;\sigma,\tau]$ is called an *elliptic number* even though it is not an elliptic (doubly periodic and meromorphic) function of a. However, $[a;\sigma,\tau]$ is a quotient of ϑ_1 functions and, as is well-known (see Whittaker and Watson [18, 21.5]), any (doubly periodic meromorphic) elliptic function can be written as a constant multiple of a quotient of products of ϑ_1 functions.

6.5. **A q-analogue of Saalschutz's summation formula.** A French mathematician Pfaff [19] discovered the summation formula

$$_3F_2\left(a,b,-n;c,1+a+b-c-n;1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \quad n\in\mathbb{N}. \tag{6.16}$$

which sums a terminating balanced $_3F_2(1)$ series with argument 1. It was rediscovered by *Saalschutz* [20] and is usually called *Saalschiitz formula* or the *Pfaff-Saalschiitz formula*; also see Askey [10]. To derive a q-analogue of equation equation (6.16) observe that

$$\frac{(abz/c;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{+\infty}\frac{(abz/c;q)_k}{(q;q)_k}\,z^k$$

The right hand side of q-analogue of Eulers tranformation formula equals

$$\sum_{k=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{(ab/c;q)_k(c/a,c/b;q)_n}{(q;q)_k(q,c;q)_m}\left(\frac{ab}{c}\right)^m z^{k+m}$$

and hence, equating the coefficients of z^n on both sides we finally get

$$ {}_3\phi_2(a,b,q^{-n},c,abc^{-1}q^{1-n};q,q)=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n},\ \ n\in\mathbb{N} \tag{6.17}$$

which was first derived by Jackson [15]. It is easy to see that equation (6.16) follows from equation (6.17) simply by replacing a,b,c in equation (6.17) by q^a,q^b and q^c respectively.

6.6. **The Bailey-Daum summation formula.** Bailey [22] and Daum [21] independently discovered the summation formula

$$ {}_2\phi_1(a,b;aq/b;q,-q/b)=\frac{(-q;q)_\infty(aq,aq^2/b^2;q^2)_\infty}{(aq/b,-q/b;q)_\infty} \tag{6.18}$$

which is a q-analogue of Kummer's formula

$$ {}_2F_1(a,b;1+a-b;-1)=\frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)}$$

The proof of The Bailey-Daum summation formula Bailey is as follows

$$\begin{aligned} {}_2\phi_1(a,b;aq/b;q,-q/b) &= \frac{(a,-q;q)_\infty}{(aq/b,-q/b;q)_\infty}\,{}_2\phi_1(q/b,-q/b;-q;q,a)\\ &= \frac{(a,-q;q)_\infty}{(aq/b,-q/b;q)_\infty}\sum_{n=0}^{+\infty}\frac{(q^2/b^2;q^2)_n}{(q^2;q^2)_n}\,a^n\\ &= \frac{(a,-q;q)_\infty}{(aq/b,-q/b;q)_\infty}\frac{(aq^2/b^2;q^2)_\infty}{(a;q^2)_\infty}\\ &= \frac{(-q;q)_\infty(aq,aq^2/b^2;q^2)_\infty}{(aq/b,-q/b;q)_\infty}\end{aligned}$$

as desired. This completes the proof of Bailey-Daum summation formula (6.18).

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