AN INTRODUCTION TO TOPOLOGICAL COMBINATORICS

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Abstract. We use Sperner's Lemma, Brouwer's Fixed Point Theorem, and The Borsuk-Ulam Theorem to prove combinatorial results.

1. Topological Theorems

1.1. **Sperner's Lemma.** The main object of Sperner's Lemma is a simplex, which is the analog of triangles in multiple dimensions. An n-simplex can be described as

$$\Delta^n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 1, x_i > 0 \right\}.$$

The vertices of this simplex are $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$. Note also that this simplex can be embedded in $\mathbb{R}^n$.

The 2-simplex as we described, and the 2-simplex embedded in $\mathbb{R}^2$ are shown in Figure 1. A 3-simplex embedded in $\mathbb{R}^3$ is shown in Figure 2

An n-simplex has $n + 1$ faces, each of which is an $n - 1$-simplex.

We are now ready to state Sperner's Lemma.

Theorem 1.1 (Sperner's Lemma). *Suppose that we have an n-dimensional polytope divided into n-simplices. Color each of the vertices of the smaller simplices one of $n+1$ colors, labeled $1, 2, \ldots, n + 1$. Call a face of a smaller n-simplex (which is an $n - 1$-simplex)* semi-full *if its vertices are colored $1, 2, \ldots, n$. Call a smaller n-simplex* full *if its vertices are colored $1, 2, \ldots, n + 1$. If N_{SF} denotes the number of semi-full faces on the exterior of the polygon and N_F denotes the number of full simplices, then $N_{SF} \equiv N_F \pmod 2$.*

One example of Sperner's Lemma is demonstrated in 2 dimensions in Figure 3, where the colors 1 2 and 3 correspond to blue red and green respectively. Note that $7 \equiv 5 \pmod 2$.

We prove this in 2 dimensions, and generalize to n dimensions from there.

Proof in 2 Dimensions. Place a dot on either side of each semi-full edge, as in Figure 4. We count the number of dots on the inside of the polygon. Each exterior semi-full edge contributes 1 dot, and each interior semi-full edge contributes 2 dots. Thus, if D denotes the number of dots on the interior of the polygon, $N_{SF} \equiv D \pmod 2$. Now, suppose that we have a triangle. If it is a full triangle, there is exactly 1 dot inside of it. If it is not, then there are 2 or 0 dots inside of it. Thus, $N_F \equiv D \pmod 2$. Hence, $N_{SF} \equiv N_F \pmod 2$. ■

It is now clear how to prove this in n dimensions.

Proof of Theorem 1.1. Place a dot on either side of each semi-full face. We count the number of dots on the inside of the polytope. Each exterior semi-full face contributes 1 dot, and each interior semi-full face contributes 2 dots. Thus, if D denotes the number of dots on the

Date: July 2021.

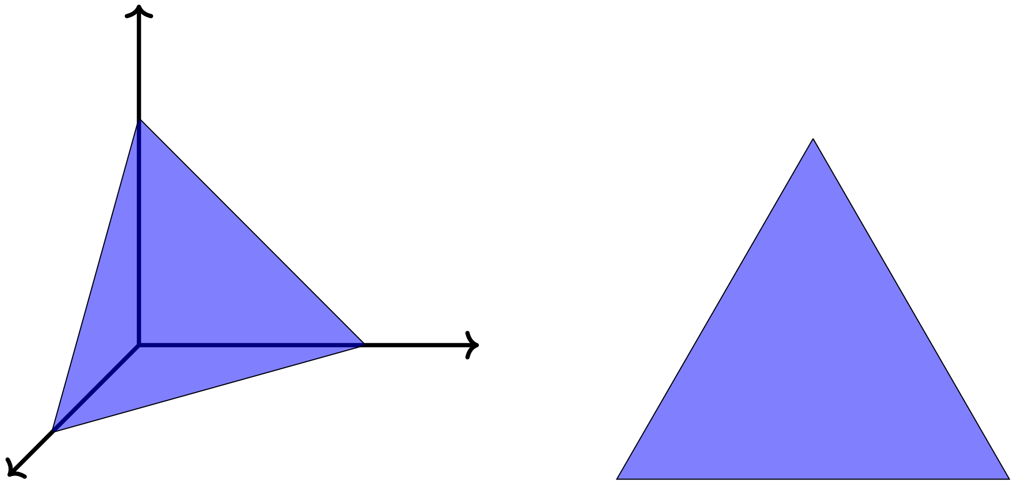


Figure 1. Δ^2 and a 2-simplex embedded in $\mathbb{R}^2$

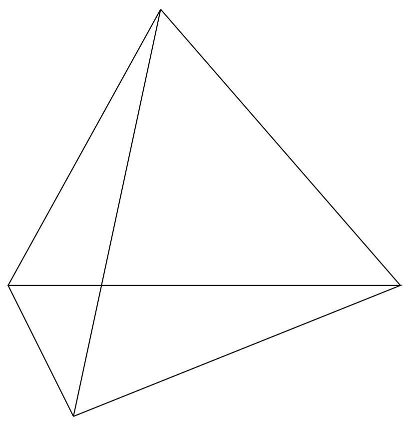


Figure 2. A 3-simplex embedded in $\mathbb{R}^3$

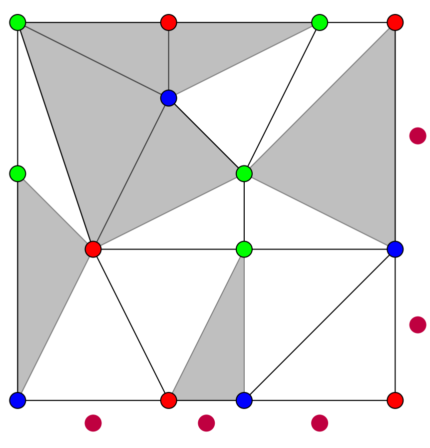


Figure 3. A case of Sperner's Lemma in 2 dimensions.

interior of the polytope, $N_{SF} \equiv D \pmod 2$. Now, suppose that we have a simplex. If it is a full simplex, there is exactly 1 dot inside of it. If it is not, then there are 2 or 0 dots inside of it. Thus, $N_F \equiv D \pmod 2$. Hence, $N_{SF} \equiv N_F \pmod 2$. ■

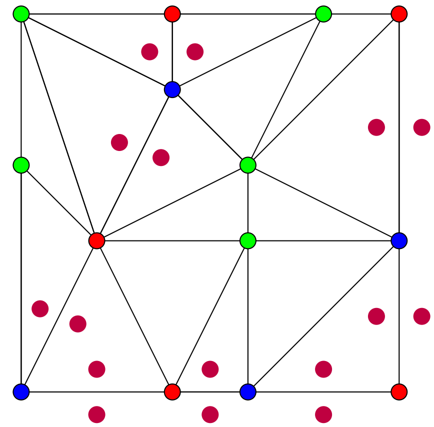


Figure 4

1.2. **The Brouwer Fixed Point Theorem.** Recall that compact sets are sets that are bounded and closed (in $\mathbb{R}^n$). Recall also that all sequences in a compact set contain a subsequence converging to an element of the set.

Proposition 1.2. *Let $K \subseteq \mathbb{R}^n$ be compact, and let $x_1, x_2, \dots$ be a sequence of points in K. Then there exists a subsequence $x_{a_1}, x_{a_2}, \dots$ converging to a point in K.*

Proof. Since K is bounded, it has finite volume. Subdivide it into 2 pieces of smaller volume. By the pigeonhole principle, at least one of these parts must contain infinitely many of the x_i's. Pick x_{a_1} in this part. Now, divide this part again in half. Once again, one of these must contain infinitely many x_i's. Pick x_{a_2} in this such that $a_2 > a_1$. Continuing this way, we get a sequence $x_{a_1}, x_{a_2}, \dots$ that eventually ends up in arbitrarily small spaces. Thus, it converges to some point $x \in \mathbb{R}^n$. As K is closed, $x \in K$. ■

Theorem 1.3 (Brouwer's Fixed Point Theorem)**.** *Let $K \subseteq \mathbb{R}^n$ be convex and compact, and let $f : K \to K$ be continuous. Then there is some $k \in K$ such that $f(k) = k$.*

Proof. We show that it suffices to check this for simplices. Let $K \subseteq \mathbb{R}^n$ be convex and compact, and let D_n denote the embedding of Δ^n into $\mathbb{R}^n$. We can easily imagine a continuous bijective function $\phi : K \to D_n$. If $f : K \to K$ has a fixed point at k, then $\phi \circ f \circ \phi^{-1} : D_n \to D_n$ has a fixed point at $\phi(k)$. Conversely, if $\phi \circ f \circ \phi^{-1}$ has a fixed point at s, then f has a fixed point at $\phi^{-1}(s)$. Thus, we only show this for simplices. Now, as topological objects, D_n and Δ^n are the same. Thus, we prove this for Δ^n. Let $f : \Delta^n \to \Delta^n$ be continuous. For $P \in \Delta^n$,

$$\sum_{i=1}^{n+1} P_i = 1 = \sum_{i=1}^{n+1} f(P)_i.$$

In particular, for every $P \in \Delta^n$, there is $1 \leq j \leq n+1$ such that $P_j \geq f(P)_j$, as we would otherwise have

$$1 = \sum_{i=1}^{n+1} P_i < \sum_{i=1}^{n+1} f(P)_i = 1.$$

If P happens to be on a k-dimensional subface (i.e. a face of a face of a face, and so on), we can choose j from among the $k+1$ nonzero coordinates. Suppose that we have a division of Δ^n into simplices, we color each vertex P the j^{th} color, where $P_j \geq f(P)_j$. We may make this into a coloring satisfying the following properties.

- Each vertex is colored differently.
- If a vertex is on a k-dimensional simplex, it is colored with one of the colors from the vertices of that simplex.

We claim that such a coloring has an odd number of semi-full faces. We prove this by induction on n. If $n = 2$, let A and B be the vertices colored 1 and 2 respectively. Then all exterior semi-full edges must fall on AB. If there are no vertices of the triangulation on AB, there are clearly an odd number of exterior semi-full edges. Suppose that the vertices $A = A_0, A_1, \ldots, A_\ell = B$ are on the edge. We go from A to B along AB, and keep track of the number of switches of color. As A and B are colored differently, this must happen an odd number of times. Now, suppose that this statement is true for $n-1$, and let $A_1, A_2, \ldots, A_n$ be the vertices of Δ^n colored $1, 2, \ldots, n$, so that all exterior semi-full faces occur on $A_1A_2\ldots A_n$. By our requirements of coloring, $A_1A_2\ldots A_n$ is a $n-1$ simplex with the appropriate coloring. Thus, there are an odd number of $(n-1)$-dimensional exterior semi-full faces. Hence, there are an odd number of $(n-1)$-dimensional full simplices. $(n-1)$-dimensional full simplices are n-dimensional semi-full faces. Thus, there exists an odd number of exterior semi-full faces.

Thus, our subdivision has an odd number of exterior semi-full faces, so it has an odd number of full simplices. In particular, there is at least one full simplex.

Let $L_1, L_2, \ldots$ be finer and finer subdivisions of Δ^n into simplices, colored as described above. Pick a full simplex T_i of L_i, and pick $x_i \in T_i$. Then the sequence $x_1, x_2, \ldots$ has a convergent subsequence $x_{a_1}, x_{a_2}, \ldots$. Let x be the limit of this sequence. We claim that x is a fixed point.

Let D_i^k denote the vertex colored i of T_{a_k}. Then, $\lim_{k\to\infty} D_i^k = x$ for all i. Now, f is continuous, so

$$f(x)_j = f\left(\lim_{k\to\infty} D_j^k\right)_j = \lim_{k\to\infty} f(D_j^k)_j \leq \lim_{k\to\infty} (D_j^k)_j = x_j.$$

We must have equality, so x is a fixed point. ■

1.3. **The Borsuk-Ulam Theorem.**

Definition 1.4. The n-dimensional sphere $\mathbb{S}^n$ is the set $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$, and the n-dimensional ball B^n is the set

$$\{x \in \mathbb{R}^n : |x| \leq 1\}.$$

The boundary of B^n is in fact $\mathbb{S}^{n-1}$, not $\mathbb{S}^n$, as $\mathbb{S}^n$ is an n-dimensional object embedded in $n+1$ dimensions, rather than an $(n-1)$-dimensional object embedded in n dimensions.

Given $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we write $-x = (-x_1, -x_2, \ldots, -x_n)$. As $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ and $B^n \subseteq \mathbb{R}^n$, this notation also applies to elements of these sets.

Theorem 1.5 (Borsuk-Ulam)**.** *Suppose that $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous. Then there is some $x \in \mathbb{S}^n$ with $f(x) = f(-x)$.*

As with Sperner's Lemma, we prove this in 2 dimensions first, and it becomes clear how to generalize this.

Proof in 2 Dimensions. Suppose that there is some function f contrary to the Borsuk-Ulam Theorem. Let $\widehat{f} : \mathbb{S}^2 \to \mathbb{R}^2$ be given by $\widehat{f}(x) = f(x) - f(-x)$. We then have that $\widehat{f} : \mathbb{S}^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$. Consider the equator α of $\mathbb{S}^2$. We claim that $\widehat{f}(\alpha)$ cannot encircle the origin.

If we slide α up to a pole, we get a nonzero point. As $\widehat{f}$ is continuous, if $\widehat{f}(\alpha)$ encircled the origin, it would cross the origin at some point. Thus, it does not encircle the origin. Now, note that $\widehat{f}(-x) = -\widehat{f}(x)$. If we split α in half, the first half must wind around the origin some $m + 0.5$ times, for $m \in \mathbb{Z}$. The second half must similarly wide around the origin $m + 0.5$ times. Thus, $\widehat{f}(\alpha)$ winds around the origin $2m + 1$ times. In particular, it winds around the origin at least once, so it does encircle the origin, a contradiction. ■

It is now clear how to generalize this.

Proof of Theorem 1.5. Suppose that the theorem is false for some f, and let $\widehat{f}(x) = f(x) - f(-x)$. Consider the equator $\mathbb{S}^{n-1}$, which we call α. Then $\widehat{f}(\alpha)$ cannot encircle the origin, as we may slide the equator to a pole (and so $\widehat{f}(\alpha)$ will go to a single nonzero point). However, $\widehat{f}(-x) = -\widehat{f}(x)$, so $\widehat{f}(\alpha)$ must contain the origin. ■

We have a name for functions satisfying $f(-x) = -f(x)$.

Definition 1.6. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be antipodal if $f(-x) = -f(x)$.

1.4. Equivalent Statements to the Borsuk-Ulam Theorem.

Theorem 1.7. *The following are equivalent.*

1. *For every continuous function $f : \mathbb{S}^n \to \mathbb{R}^n$, there is some $x \in \mathbb{S}^n$ with $f(x) = f(-x)$.*
2. *For every continuous antipodal function $f : \mathbb{S}^n \to \mathbb{R}^n$, there is some $x \in \mathbb{S}^n$ with $f(x) = 0$.*
3. *There does not exist a continuous, antipodal mapping $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$.*
4. *There does not exist a continuous mapping $f : B^n \to \mathbb{S}^{n-1}$ that is antipodal on the boundary of B^n.*

Proof. We show that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (2) Suppose that $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous and antipodal. Then there is some $x \in \mathbb{S}^n$ with $f(x) = f(-x) = -f(x)$. Thus, $f(x) = 0$.

(2) $\Rightarrow$ (1) Suppose $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous. Then $\widehat{f}(x) = f(x) - f(-x)$ is antipodal, so there is some $x \in \mathbb{S}^n$ with $f(x) - f(-x) = 0$. Thus, $f(x) = f(-x)$.

(2) $\Rightarrow$ (3) Suppose that there exists a continuous antipodal mapping $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$. Then there exists a continuous antipodal mapping to $f : \mathbb{S}^n \to \mathbb{R}^n \setminus \{\mathbf{0}\}$, a contradiction.

(3) $\Rightarrow$ (2) Suppose that $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous and antipodal, and is nonzero everywhere. Then, $\widehat{f}(x) = \frac{f(x)}{|f(x)|}$ is a continuous antipodal mapping from $\mathbb{S}^n$ to $\mathbb{S}^{n-1}$.

(3) $\Rightarrow$ (4) Suppose that there is a continuous function $f : B^n \to \mathbb{S}^{n-1}$ that is antipodal on the boundary of B^n. Define $\pi : B^n \to \mathbb{S}^n$ by

$$(x_1, x_2, \ldots, x_n) \mapsto \left(x_1, x_2, \ldots, x_n, \sqrt{1 - x_1^2 - x_2^2 - \cdots - x_n^2}\right).$$

Notice that π is continuous and, restricting its image to the upper hemisphere of $\mathbb{S}^n$, bijective. Write $f = g \circ \pi^{-1}$. Then, f is continuous and antipodal on the equator of $\mathbb{S}^n$. We can extend f to a continuous antipodal function from $\mathbb{S}^n$ to $\mathbb{S}^{n-1}$.

(4) $\Rightarrow$ (3) Suppose that there is a continuous antipodal mapping $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$. Define $g = f \circ \pi$. Then $g : B^n \to \mathbb{S}^{n-1}$ is continuous and antipodal on the boundary of B^n.

■

Note that (1) is the Borsuk-Ulam Theorem.

1.5. **Topological Consequences of the Borsuk-Ulam Theorem.** We provide an alternate proof of the Brouwer Fixed Point Theorem, using the Borsuk-Ulam Theorem.

Alternate Proof of Theorem 1.3. For the exact reason that it suffices to show this theorem for simplices, it suffices to show this theorem for B^n. Suppose that $f : B^n \to B^n$ has no fixed points. We construct $g : B^n \to \mathbb{S}^{n-1}$ by choosing $g(x)$ to be the intersection of $\mathbb{S}^{n-1}$ and the ray from $f(x)$ to x. This is well defined and the identity on the boundary of B^n. In particular, it is antipodal on the boundary. It is clearly possible to express $g(x)$ in terms of continuous functions and $f(x)$, which is continuous, so $g(x)$ is continuous, a contradiction. ■

We can also prove a theorem regarding coverings of $\mathbb{S}^n$.

Theorem 1.8 (Lyusternik-Shnirel'man)**.** *Suppose that $U_1, U_2, \ldots, U_{n+1}$ are sets covering $\mathbb{S}^n$. Suppose further that U_i is either open or closed for $1 \leq i \leq n$. Then one of the U_i contains a pair of antipodal points.*

Proof. Suppose that none of the U_i contains a pair of antipodal points. Define

$$\delta(x, U_i) = \inf_{u \in U_i} |x - u|$$

to be the distance from x to U_i. Define a map from $\mathbb{S}^n$ to $\mathbb{R}^n$ by

$$x \mapsto \big(\delta(x, U_1), \delta(x, U_2), \ldots, \delta(x, U_n)\big).$$

This is continuous, so there is some x with $f(x) = f(-x)$. At least one of x and $-x$ is not in U_{n+1}, by assumption. Suppose without loss of generality that this is x. Choose i such that $x \in U_i$. Then, $\delta(x, U_i) = 0 = \delta(-x, U_i)$.

Suppose that U_i is closed. Then, for $\varepsilon > 0$, there is some $u_\varepsilon \in U_i$ with $|u_\varepsilon - (-x)| < \varepsilon$. Then, $u_1, u_{1/2}, u_{1/3}, \ldots$ is a sequence converging to $-x$ in U_i, so $-x \in U_i$, a contradiction.

Suppose that U_i is open. By the previous argument, we have that $-x \in \overline{U_i}$. Now, by assumption, U_i does not contain a pair of antipodal points, so $U_i \subseteq \mathbb{S}^n \setminus -U_i$. The latter set is closed, so $\overline{U_i} \subseteq \mathbb{S}^n \setminus -U_i$. Thus, $-x \in \overline{U_i} \subseteq \mathbb{S}^n \setminus -U_i$, so that $-x \notin -U_i$. Thus, $x \notin U_i$, a contradiction. ■

2. Basic Combinatorial Results

2.1. **Tucker's Lemma.**

Theorem 2.1 (Tucker's Lemma)**.** *Let T be a subdivision of B^n into simplices that is antipodally symmetric on the boundary. Write $V(T)$ for the set of vertices of T. Let $\chi : V(T) \to \{\pm 1, \pm 2, \ldots, \pm n\}$ be a coloring satisfying $\chi(-v) = -\chi(v)$ for v on the boundary of B^n. Then there exists an edge of T which has vertices labeled by opposite numbers.*

We reformulate this to make it easier to prove. We write Δ for a collection of simplices with vertices $V(\Delta) = \{\pm 1, \pm 2, \ldots, \pm n\}$. We say $F \subseteq V(\Delta)$ forms a simplex if there is no $1 \leq i \leq n$ with $i, -i \in F$. We now have the following equivalent theorem to Tucker's Lemma.

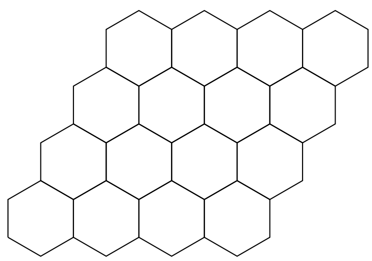
Theorem 2.2. *Let T be a subdivision of B^n into simplices that is antipodally symmetric on the boundary. Then, there is no map $\chi : V(T) \to V(\Delta)$ that is antipodal on the boundary and maps simplices of T to simplices of Δ.*

Proof. Suppose that there is such a map $\chi : V(T) \to V(\Delta)$. We describe how to construct a continuous map from B^n to $\mathbb{S}^{n-1}$ that is antipodal on the boundary. Let $x \in B^n$, and let the vertices of an n-simplex containing x (there may be more than one) be $A_1, A_2, \ldots, A_{n+1}$. Now, there are unique $a_1, \ldots, a_{n+1}$ with $\sum_{i=1}^{n+1} a_i = 1$ and $x = \sum_{i=1}^{n+1} a_i A_i$. Let $\pi : \{\pm 1, \pm 2, \ldots, \pm n\} \to \mathbb{R}^n$ be given by $\pi(1) = (1, 0, \ldots, 0)$, $\pi(-1) = (-1, 0, \ldots, 0)$, $\pi(2) = (0, 1, \ldots, 0)$, $\pi(-2) = (0, -2, \ldots, 0)$, and so on. Then, the map

$$x \mapsto \sum_{i=1}^{n+1} a_i (\pi \circ \chi)(A_i)$$

is well defined and continuous. Further, it is antipodal on the boundary. ■

2.2. **Hex.** The game of Hex is played between two players, red and green. Each of them takes turns coloring one hexagon in a grid. An example of a possible grid is shown below. After all hexagons are filled, red wins if there is a red path connecting the top and bottom, and green wins of there is a green path connecting the left and right.



We show that this game cannot end in a draw. We first change the hexagons into vertices and the boundaries between hexagons to edges. We then have to color the vertices.

Suppose that there is a coloring that translates to a drawn hex game. Write R_0 for the red vertices reached from the bottom by a red path, R_1 for the remaining vertices, G_0 for the green vertices reached from the left by a green path, and G_1 for the remaining green vertices. Define e_1 to be a rightward shift by 1 vertex parallel to the top and bottom sides, and e_2 to be an upward shift by 1 vertex parallel to the left and right sides. Define

$$f(v) = \begin{cases} v + e_1 & v \in G_0 \\ v - e_1 & v \in G_1 \\ v + e_2 & v \in R_0 \\ v - e_2 & v \in R_1 \end{cases}.$$

No vertex goes off of the board, as we assumed that the game is drawn. Suppose that we have a triangle with vertices v_1, v_2, v_3. Then, each x in the triangle can be written uniquely as $x = \sum x_i v_i$, where $x_i \geq 0$ and $\sum x_i = 1$. We can then define $f(x) = \sum x_i f(v_i)$, to get a continuous map. Thus, there is a fixed point $x = \sum x_i v_i$, for some triangle with vertices v_1, v_2, v_3. Let $\varepsilon_i \in \{\pm e_1, \pm e_2\}$ so that $f(v_i) = v_i + \varepsilon_i$. We thus have that $\sum x_i \varepsilon_i = 0$. Now, as 0 is not a fixed point of f, $x \neq 0$, so some $x_i > 0$, say x_1. Then, at least one of $\varepsilon_2, \varepsilon_3$ must be $-\varepsilon_1$. However, this is a contradiction, as elements of G_0 (R_0) and G_1 (R_1) cannot be connected, and so cannot be part of the same triangle.

2.3. **Necklace Division.** Suppose that two thieves steal a necklace with n different types of gems on a string, with an even number of each type of gem. Suppose further that they wish to cut up the necklace and divide the pieces so that each person receives an equal amount of each gem.

Theorem 2.3. *Suppose that a necklace with n different types of gems on a string is to be split equally between two people by cutting it into separate pieces and distributing them. Then, this can be done in n cuts.*

Proof. Let us allow ourselves to cut gems into pieces. Scale the necklace to lie in $[0,1]$, and cut it into intervals of lengths $c_1, c_2, \ldots, c_{n+1}$. We define

$$a_i = \begin{cases} \sqrt{c_i} & c_i \text{ is given to person 1} \\ -\sqrt{c_i} & c_i \text{ is given to person 2} \end{cases}.$$

Then, $(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{S}^n$. Let b_i be the number of gems of type i that the first person gets. We define $f : \mathbb{S}^n \to \mathbb{R}^n$ by

$$(a_1, a_2, \ldots, a_{n+1}) \mapsto (b_1, b_2, \ldots, b_n).$$

This is clearly continuous, so we may apply the Borsuk-Ulam Theorem to find $x \in \mathbb{S}^n$ with $f(x) = f(-x)$. Now, swapping the signs of the a_i swaps the amounts that the two people have, so we have a solution if we allow ourselves to cut gems.

Suppose that our solution cuts some gem of type i. Then, it must cut at least one more gem of type i. Let a be the total number of gems that the first person receives from the cut up gems, and let b be the total number of gems that the second person receives from the cut up gems. Then $a, b \in \mathbb{N}$, as each person gets a whole number of gems. We may move the cuts to give person 1 a whole gems, and to give person 2 b whole gems. We may do this for all i to obtain a division that does not cut gems. ■

3. Kneser's Conjecture

One of the first theorems in combinatorics proved using topological methods is Kneser's Conjecture. To discuss this, we need to review some graph theory. Rather than Lovász's original proof, we provide a simpler topological proof by Greene.

3.1. Graph Theory.

Recall that a graph is a pair $G = (V, E)$ with $E \subseteq \binom{V}{2}$.

Definition 3.1. Given a graph $G = (V, E)$, its chromatic number $\chi(G)$ is the smallest number $k \in \mathbb{N}$ such that there exists a function $c : V \to [k]$ satisfying $c(v_1) \neq c(v_2)$ when $\{v_1, v_2\} \in E$.

Definition 3.2. Given positive integers n, k with $n \geq 2k$, the Kneser graph is $\mathrm{KG}(n, k)$, which has

$$V = \binom{[n]}{k}$$
$$E = \big\{\{e, f\} : e, f \in V, e \cap f = \varnothing\big\}.$$

We have that $\mathrm{KG}(n, 1)$ is the complete graph on n vertices K_n. We also have that $\mathrm{KG}(2k, k)$ consists of $\binom{2k}{k}/2$ pairs of vertices. The easiest graph not of these two forms is $\mathrm{KG}(5, 2)$, which is shown in Figure 5.

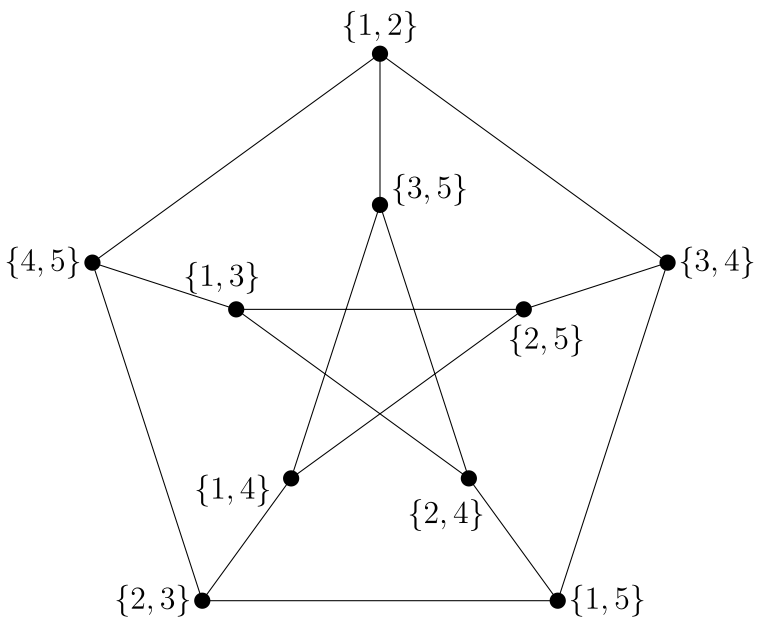


Figure 5. $K(5,2)$

3.2. **Kneser's Conjecture.** Kneser's Conjecture investigates the chromatic number of the Kneser graph. More precisely, we have the following.

Theorem 3.3. *The chromatic number of the Kneser graph is* $\chi(\mathrm{KG}(n,k)) = n-2k+2$.

Proof. We first provide a coloring of $\mathrm{KG}(n,k)$ using $n-2k+2$ colors. Define $c : \binom{[n]}{k} \to [n-2k+2]$ by

$$u \mapsto \min\big\{\min\{x : x \in u\}, n-2k+2\big\}.$$

We show that two vertices receiving the same color are not connected. Suppose that $u, v \in \binom{[n]}{k}$ have $c(u) = c(v) = c$. If $c < n-2k+2$, then $c \in u \cap v$. If $c = n-2k+2$, then $u, v \subseteq \{n-2k+2, \ldots, n\}$. However, this contains $2k-1$ elements, so u and v cannot be disjoint. Thus, $\chi(\mathrm{KG}(n,k)) \le n-2k+2$.

Now, suppose that $\chi(\mathrm{KG}(n,k)) < n-2k+2$. Let $c : \binom{[n]}{k} \to [n-2k+1]$ be a coloring. Write $d = n-2k+1$. Choose a set X of n vectors on $\mathbb{S}^d$ such that any $d+1$ of the vectors are linearly independent. Identify these vectors with $[n]$. Then, each vertex of $\mathrm{KG}(n,k)$ corresponds to a set of k vectors on the sphere. Given $x \in \mathbb{S}^d$, write

$$H(x) = \{y \in \mathbb{S}^d : x \cdot y > 0\}$$

for the open hemisphere with pole x. For $1 \le i \le d$, let

$$U_i = \big\{x \in \mathbb{S}^d : \exists S \subseteq X, |S| = k, c(S) = i, S \subseteq H(x)\big\},$$

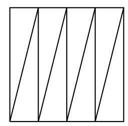
which is clearly open. Let $A = \mathbb{S}^d \setminus (U_1 \cup \cdots \cup U_d)$. We show that none of these sets contains a pair of antipodal points, obtaining a contradiction.

Suppose that $x \in U_i$, so that $H(x)$ contains a subset of X of order k colored with color i. $H(x)$ and $H(-x)$ are disjoint, so $H(-x)$ cannot contain a subset of X of order k colored i. If it did, this would give $u, v \in \binom{[n]}{k}$ with $u \cap v = \varnothing$ and $c(u) = c(v) = i$. Thus, $-x \notin U_i$.

Now, suppose that $\pm x \in A$. By definition, neither one of $H(x)$ and $H(-x)$ contains a subset of X of order k. Thus, there are at least $n-2(k-1) = n-2k+2 = d+1$ points of X lying on the equator (with respect to x) $\{y \in \mathbb{S}^2 : x \cdot y = 0\}$. This is contained in a subspace of dimension d, which contradicts the fact that any $d+1$ vectors of X are linearly independent. ∎

4. Monsky's Theorem

4.1. **Dividing Squares.** Suppose that we wish to divide a square into n triangles of equal area. If $n = 2k$, this is simple. We divide the square into k rectangles, and cut each rectangle in half, as below.



However, when n is odd, this problem is not as clear. It turns out that this is impossible. To do this, we need to discuss norms.

4.2. **Norms.** Consider the absolute value from $\mathbb{R}$ to $\mathbb{R}$, which has

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

Let us look at the most important properties of this function. Firstly, $|x| \geq 0$ for all x, and $|x| = 0$ if and only if $x = 0$. We also have that $|xy| = |x| \cdot |y|$ and $|x+y| \leq |x| + |y|$.

Definition 4.1. A norm on $\mathbb{R}$ is a function $|\cdot| : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

- $|x| = 0$ if and only if $x = 0$
- $|xy| = |x| \cdot |y|$
- $|x+y| \leq |x| + |y|$.

Proposition 4.2. *If $|\cdot|$ is a norm, then $|-1| = |1| = 1$, so that $|-a| = |a|$ for any $a \in \mathbb{R}$.*

Proof. For $x \neq 0$, $|1| \cdot |x| = |1 \cdot x| = |x|$, so $|1| = 1$. We also have that $|-1|^2 = |-1 \cdot -1| = |1| = 1$, so $|-1| = \pm 1$. As $|-1| \geq 0$, $|-1| = 1$. ∎

Norms need not only be defined on $\mathbb{R}$. For example, the p-adic norm $|\cdot|_p$ on $\mathbb{Q}$ for prime p is defined as

$$\left|p^k \frac{a}{b}\right|_p = p^{-k} \qquad p \nmid a, b$$

and $|0|_p = 0$. It is clear that this is a norm, as it satisfies the three requirements in the definition. This norm in fact satisfies a stronger condition than the triangle inequality, known as the ultrametric triangle inequality.

Proposition 4.3. *For any $r, s \in \mathbb{Q}$, we have $|r+s|_p \leq \max(|r|_p, |s|_p)$.*

Proof. Write $r = p^k \frac{a}{b}$ and $s = p^\ell \frac{c}{d}$, where $p \nmid a, b, c, d$. Without loss of generality, suppose $k \geq \ell$, so that $|r|_p \leq |s|_p$. Then,

$$\begin{aligned} |r+s|_p &= \left|p^k \frac{a}{b} + p^\ell \frac{c}{d}\right|_p = \left|p^\ell \left(p^{k-\ell}\frac{a}{b} + \frac{c}{d}\right)\right|_p = p^{-\ell} \left|\frac{p^{k-\ell}ad + bc}{bd}\right|_p \\ &= p^{-\ell} |p^{k-\ell}ad + bc|_p \leq p^{-\ell} = |s|. \end{aligned}$$

∎

We also have a form of equality.

Proposition 4.4. *If $|\cdot|$ is a norm satisfying the ultrametric triangle inequality, and $|r| \neq |s|$, then $|r+s| = \max(|r|, |s|)$.*

Proof. Suppose without loss of generality that $|r| > |s|$, and suppose that $|r+s| < |r|$. Then,

$$|r| = |(r+s) - s| \leq \max(|r+s|, |s|) < |r|.$$

■

Now, the proof of our theorem relies on the following important fact.

Theorem 4.5. *There exists a norm $|\cdot|$ on $\mathbb{R}$ satisfying the ultrametric triangle inequality such that $|\frac{1}{2}| = 2$.*

This can be thought of as "extending" the 2-adic norm from $\mathbb{Q}$ to $\mathbb{R}$. We provide a sketch of the proof of Theorem 4.5. In order to do so, we need Zorn's Lemma.

Theorem 4.6 (Zorn's Lemma)**.** *Let P be a poset such that, for any totally ordered $T \subseteq P$, there is some $a \in P$ with $t \leq a$ for all $t \in T$. Then there is some $p \in P$ such that, if $p \leq q$, $p = q$.*

Sketch of Proof of Theorem 4.5. Consider the pairs $(A, |\cdot|)$, where A is a subring of $\mathbb{R}$ containing $\frac{1}{2}$, and $|\cdot|$ is a norm on A satisfying the ultrametric triangle inequality with $|\frac{1}{2}| = 2$. We make this into a poset by defining $(A, |\cdot|_A) \leq (B, |\cdot|_B)$ when $A \subseteq B$ and $|a|_A = |a|_B$ for all $a \in A$. Suppose that we have a totally ordered subset $\big\{(A_i, |\cdot|_{A_i}) : i \in I\big\}$. Define $B = \bigcup_{i \in I} A_i$ and $|a|_B = |a|_{A_i}$ whenever $a \in A_i$. This is well defined, and we have that $(B, |\cdot|_B) \geq (A_i, |\cdot|_{A_i})$ for all i. Thus, the conditions of Zorn's Lemma are satisfied, so there exists a maximal element.

Suppose that $(A, |\cdot|)$ is a maximal element. We claim that $A = \mathbb{R}$. If not, take $\alpha \in \mathbb{R} \setminus A$. Let B be the smallest subring of $\mathbb{R}$ containing both A and α. Suppose that α is the root of a nonzero polynomial with coefficients in A. It is possible to extend $|\cdot|$ to B, but this relies on results beyond the scope of this paper. If α is not the root of a polynomial with coefficients in A, we can choose $|\alpha|$ to be whatever we want. Pick $|\alpha| = 1$. We then have $|a\alpha^n| = |a|$ for $a \in A$. If $p(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0$, we define $|p(\alpha)| = \max_{0 \leq i \leq n} |a_i|$. It is easy to check that this defines a norm on B. ■

4.3. **Monsky's Theorem.** Throughout this section, we use $|\cdot|$ to refer to a norm which exists by Theorem 4.5.

Theorem 4.7 (Monsky)**.** *It is impossible to dissect a square into an odd number of triangles with equal area.*

Suppose that we have a square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. We construct a coloring that we can apply Sperner's Lemma to. We color a point (x, y)

- blue if $|x| \geq |y|$ and $|x| \geq 1$
- green if $|x| < |y|$ and $|y| \geq 1$
- red if $|x| < 1$ and $|y| < 1$,

where blue, red and green correspond to colors 1, 2 and 3 respectively. Notice that every point is colored: if (x, y) is not colored blue or green, then

$$\big(|x| < |y| \vee |x| < 1\big) \wedge \big(|x| \geq |y| \vee |y| < 1\big).$$

If $|x| < |y|$, then $|y| < 1$, so $|x| < 1$ and (x, y) is red. If $|x| < 1$ and $|x| \geq |y|$, then $|y| < 1$, so (x, y) is red. If $|x| < 1$ and $|y| < 1$, then (x, y) is red.

Lemma 4.8. *Suppose that we have a full triangle ABC. Then its area $[ABC]$ satisfies $\big|[ABC]\big| \geq 2$.*

Proof. Suppose that $A = (x_1, y_1)$ is red, $B = (x_2, y_2)$ is green, and $C = (x_3, y_3)$ is blue. We claim that $(x_2 - x_1, y_2 - y_1)$ is green and $(x_3 - x_1, y_3 - y_1)$ is blue. We have that $|x_2 - x_1| \leq \max(|x_2|, |x_1|)$. If $|x_2| > |x_1|$, then $\max(|x_2|, |x_1|) < |y_2|$. If $|x_1| \geq |x_2|$, then $\max(|x_2|, |x_1|) < 1 \leq |y_2|$. Now, $|y_2| = \max(|y_2|, |y_1|) = |y_2 - y_1|$ ($|y_2| \neq |y_1|$). We also have that $|y_3 - y_1| \leq \max(|y_3|, |y_1|)$. If $|y_3| > |y_1|$, then $\max(|y_3|, |y_1|) \leq |x_3|$, and if $|y_1| \geq |y_3|$, then $\max(|y_3|, |y_1|) \leq |x_3|$. Now, $|x_3| = \max(|x_3|, |x_1|) = |x_3 - x_1|$ ($|x_3| \neq |x_1|$). Thus, $B - A$ is green and $C - A$ is blue. We thus assume $A = (0, 0)$, as this is red.

We have that the signed area of ABC is

$$[ABC] = \frac{x_2 y_3 - x_3 y_2}{2}.$$

Thus,

$$\big|[ABC]\big| = \frac{|x_2 y_3 - x_3 y_2|}{|2|} = 2|x_2 y_3 - x_3 y_2|.$$

We have that $|x_2| < |y_2|$ and $|x_3| \geq |y_3|$, so $|x_2 y_3| < |x_3 y_2|$. Thus,

$$|x_2 y_3 - x_3 y_2| = |x_3 y_2| = |x_3| \cdot |y_2| \geq 1.$$

Thus, $\big|[ABC]\big| \geq 2$. ■

If n is odd, then we show that $|1/n| = 1$. Notice that $|2^k| = 2^{-k}$, as $2^k = (1/2)^{-k}$. Now, write $n = 2^0 + a_1 2^1 + \cdots + a_\ell 2^\ell$, where $a_1, a_2, \ldots, a_\ell \in \{0, 1\}$. We then have that

$$|n| = \max\left\{1, \frac{a_1}{2}, \frac{a_2}{2^2}, \ldots, \frac{a_\ell}{2^\ell}\right\} = 1.$$

Thus, $|1/n| = 1$. Thus, if we have a division of the square into n triangles of equal area, none of the triangles can be full triangles. Monsky's Theorem is established by the following lemma.

Lemma 4.9. *Any dissection of the square into finitely many triangles must contain a full triangle.*

Proof. Note that $(0, 0)$ is red, $(1, 0)$ is blue, $(1, 1)$ is blue, and $(0, 1)$ is green. Further, along $y = 0$, every point is either red or blue, along $x = 0$, every point is either red or green, and along $x = 1$ and $y = 1$, every point is blue or green. Thus, all exterior semi-full edges lie on $y = 0$. Walking from $(0, 0)$ to $(1, 0)$, we start at a red vertex and end at a blue vertex, so we switch colors an odd number of times. Hence, there are an odd number of exterior semi-full edges. By Sperner's Lemma, there are an odd number of full triangles, and so at least one full triangle. ■