AN INTRODUCTION TO TOPOLOGICAL COMBINATORICS

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Abstract. In this paper we go through a few application of topology in combinatorics. We will see how combinatorial problems can be elegantly solved using topological methods like the Borsuk-Ulam Theorem and Brouwer's Fixed Point Theorem.

1. Introduction

In 1978, Lovász proved Kneser's Conjecture and this proof marked the birth of a new field in mathematics called topological combinatorics. By using topological concepts like the Borsuk-Ulam Theorem, Lovász was able to prove a combinatorial problem. After this, there have been many applications of topology in combinatorics and in this paper we will go over a few.

2. Basic Topology

Before we go on, we will need to have to know the basics of topology. This paper will not go into to too much depth since the goal is to study the applications of the field. Topology is the study of geometric objects under continuous deformations: this includes stretching, squishing, folding, or any other transformation that does not cut the object. The motivation of topology is to study objects not based on their actual shape but more general characteristics. We start topology by defining our space

Definition 2.1. If X is some set and τ is a set of subsets of X, the ordered pair (X, τ) is considered *topological space* if three properties are satisfied

1. The empty set and X are in τ
2. The union of any number elements of τ is in τ
3. The intersection of any number elements of τ is in τ

Additionally, for any topological space (X, τ), τ is said to be a *topology* on X.

Now that we have defined our space, we can start defining concepts in topology. One concept we will be using is called homotopy.

Definition 2.2. Let X and Y be topological spaces and f and g be functions from X to Y. A function $H : X \times [0, 1] \to Y$ is a *homotopy* between f and G if

$$H(x, 0) = f(x), H(x, 1) = g(x)$$

for all $x \in X$.

There is an conceptual way of thinking about this definition by looking at $[0, 1]$ as a continuous time interval. If we have two continuous functions and we can deform one function into the other function over this time interval, the deformation is called a *homotopy* and the

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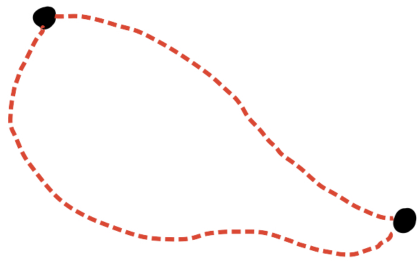

FIGURE 1. Homotopic Paths

condition $H(x, 0) = f(x), H(x, 1) = g(x)$, is the just the initial conditions for the deformation. For example, if we have two paths between two points, which are functions, we can stretch and squish one path to form the other, the paths are homotopic and the way we streched and squished the path is a homotopy. This can be seen in Figure 1.

Definition 2.3. For a given function f, its homotopy class is the set of all functions that are homotopic to f.

Now we us define homotopy groups.

Definition 2.4. If b_1 is the base point of the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ and b_2 is the base point of some topological space X, the nth homotopy group $\pi_n(X)$ is the set of homotopy classes of the functions

$$f : S^n \to X$$

that map b_1 to b_2.

Using this, we can define n-connectedness.

Definition 2.5. A topological space X is n-connected if its first n homotopy groups are trivial. Connectedness is denoted with $\text{conn}(X)$.

We are done with our definitions now but any field is pointless without theorems that use the definitions so one of the topological theorems that we will be using is the Borsuk-Ulam Theorem.

Theorem 2.6 (Borsuk-Ulam). *If the function $f : S^n \to \mathbb{R}^n$ is continuous, there exists $x \in S^n$ such that $f(x) = f(-x)$.*

Corollary 2.7. *There are always a pair antipodal points on Earth with exactly the same temperature and pressure.*

Proof. Temperature and Pressure are functions $f : S^2 \to \mathbb{R}^2$ and applying 2.6 gives us $f(x) = f(-x)$ which are antipodal points. ■

Another theorem we will using is Brouwer's Fixed Point Theorem.

Theorem 2.8 (Brouwer's Fixed Point Theorem). *If D^2 is a disk in $\mathbb{R}^2$ and $f : D^2 \to D^2$ is a continuous function, f always has a fixed point i.e. there always exists a point $x \in D^2$ such that $f(x) = x$.*

3. Proof of the Kneser Conjecture

As stated in the introduction, Lovász's proof was the first application of topological combinatorics so we will be starting with this. First we define chromatic numbers.

Definition 3.1. Let the *chromatic number* of a graph $\chi(G)$ is the least number of colors needed to color G such that adjacent vertices are not the same color.

From now on, let colarable meant that adjacent vertcies are not the same color.

Definition 3.2. A *Kneser Graph* $K(n, k)$ is a graph where vertices are k element subsets of $[n]$ and are connected with an edge if the subsets are disjoint.

Now the Kneser Conjecture is

Conjecture 3.3 (Kneser)**.**

$$\chi(K(n, k)) = n - 2k + 2.$$

Before we use topology we need to define a few things. For a graph G, let $\mathcal{N}(G)$ be the set of a vertices that share a neighbor. This is known as the neighborhood complex. Note that $\mathcal{N}(G)$ is topological space since it satisfies all three conditions. Now we prove this proposition.

Proposition 3.4. *For any graph G,*

$$\chi(G) = conn(\mathcal{N}(G)) + 3.$$

Then since $\mathcal{N}(K(n, k))$ is $n + k - 1$ connected, from this proposition, we can prove the Kneser Conjecture. All that is left is to prove the proposition. Before we go on, we state the Borsuk-Ulam Theorem in another way.

Corollary 3.5. *If there is an antipodal continuous map $f : S^n \to S^m$, then $m \geqslant n$.*

Proof 3.4. First let $\text{conn}(\mathcal{N}(G)) = k$. Now if G is m colorable, this means that there is a *graph homomorphism* $G \to K_m$ from G to the complete graph K_m. This roughly means that there is a map from G tp K_m that maintains the structure. This must mean that the graph homomorphism $\mathcal{N}(G) \to \mathcal{N}(K_m)$ also exists. With this and the fact that $\text{conn}(\mathcal{N}(G)) = k$, we can construct the an antipodal continuous map $f : S^{k+1} \to S^{m-2}$. By our new Borsuk Ulam Theorem, we have

$$m - 2 \geqslant k + 1 \implies m \geqslant k + 3.$$

Therfore the minimum number of colors needed color G or $\chi(G)$ is $k + 3 = \text{conn}(\mathcal{N}(G)) + 3$. ■

4. The Necklace Problem

Another application of the Borsuk-Ulam Theorem is in the Necklace Problem.

Proposition 4.1. *Let there be $2d > 2$ jewels on a string where each jewel is one of n types. It is always possible to use n or fewer cuts to cut and divide the substrings among two people where each person gets the same number of jewels of each type.*

An example of this proposition is shown in Figure 2.

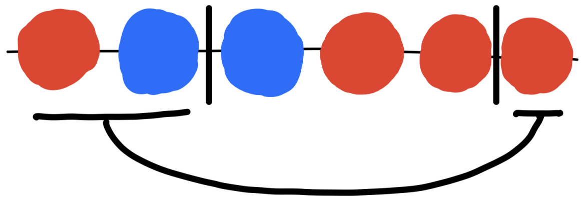

FIGURE 2. Example of cutting and division of a necklace

Proof. Our first step is to convert our problem into a continuous one. First we put our string, say of length 1, on a number line and split our string into $2d$ regions. Then we color each region depending on the jewel type in that region. Now the way we cut our string is by picking numbers between 0 and 1. If our cut is in the middle of a "jewel region", we just round down to the next highest "jewel region."

Now that we have a continuous problem, we can start using the Borsuk-Ulam Theorem. The way we do this is by first creating a bijection betwen points on a n-sphere and ways of cutting and dividing the jems. Recall that the n-sphere is all points $(x_1, x_2, \cdots, x_{n+1})$ such that

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1.$$

We can encapculate this information in our string of length 1 with n jewels. Cutting with n cuts is equivilant to finding $n+1$ numbers $x_1^2, x_2^2, \cdots, x_{n+1}^2$. Then the sign of the square root tells us which person to give the jewels to. Therefore every point on a n-sphere can be uniquely created into a cutting and division of string of length 1 with n jewels.

Now let $f : S^n \to \mathbb{R}^n$ be a continuous function and S^n is a n-sphere. Using what we got above, f is a continuous function that maps a cutting and division of a string of length 1 with n jewels to $\mathbb{R}^n$. Now let a point in $\mathbb{R}^n$ represent the number of jewels the first person gets of each type. Now by the Borsuk-Ulam Theorem, there must exist a point $f(x) = f(-x)$. If x, a point on the n-sphere, is some cutting and division, $-x$ is the same cutting but we flip who get's each substring we cut. Now $f(x)$ is the number of jewels of each type the first person gets, and $f(-x)$ is the number of jewels of each type the first person would get if the divisions were flipped among the people or the number of jewels of each type the second person currently has. Therefore $f(x) = f(-x)$ says that the number of jewels of each type the first person gets is the same as the number of jewels of each type the second person gets. Therefore there always exists a fair division. ■

5. INSCRIBED RECTANGLE PROBLEM

A fairly famous unsolved problem is the inscribed square problem:

Question 5.1. *Can we always inscribe a square in any loop?*

However a weaker version of this problem that we can prove is the inscribed rectangle problem:

Proposition 5.2. *It is always possible to inscribe a rectangle in any loop.*

Proof. We can start this proof by noticing what need to form a rectangle. Rather than using the orthodox method of defining a rectangle as a parallelogram with all right angles, we use the fact that if we find two line segments that share a midpoint and are the same length, the

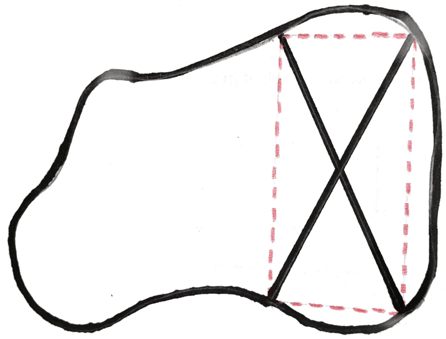

Figure 3. A rectangle inscribed in a loop

for points form a rectangle. Therefore our goal is to find two inscribed line segments such that they share a midpoint and are the same length. We can see an example in Figure 3. Another way of thinking of our goal is by first placing the loop on the xy-plane. Then for any inscribed line segment, we look at it's midpoint and we plot a point in $\mathbb{R}^3$ with x and y coordinates as the midpoint and a z coordinate as the length of the line segment. What we will end up with is a surface above the loop and we must prove that it intersects itself.

To do this we visualize pairs of points differently. First we turn our loop into a line sgement. Now we "plot" pairs of points by tilting our line segment 90 degrees and plotting like Cartesian coordinates inside the square formed by the line segments. Now our plot is not fully completed since we glue together edges that represent the same point. Namely opposite sides of the square need to be glued and all pairs of points that are symmetric about the NE diagonal need to be put together. By folding and gluing, we get the Möbius Strip. That this means is that every point on the Möbius Strip represents a pair of points on our loop. Also, note that the edge of Möbius Strip is when the pair of points are the same.

Now that we have this representation, since pairs of points define line segments, we must now be able to map the Möbius Strip onto our surface. However, we have a restriction: the edge of the Möbius Strip must map onto the loop. This is because the edge of the Möbius Strip represents all points on the loop so this must map onto the part of the surface that is the loop. Now we see that when we map the Möbius Strip, because of its nature and how it is formed by twisting a strip, there must always be some point where the surface intersects itself. This in turn means that there are two line segments that have the same midpoint and have the same length so these four points make our rectangle. Therefore we are done. ∎

6. Hex Game

Topology can also be used in analyzing games. Consider the game Hex with the following simple rules

1. The game is played on a rhombus game board with pairs of opposite sides colored red and blue and players are assigned to these opposite sides.
2. The players take turns coloring cells in the game board with their color.
3. Whichever player connects their two opposite sides with a path of their color wins.

An example game could look like Figure 4 where red has won.

Theorem 6.1 (Hex Theorem)**.** *Hex can never end in a draw.*

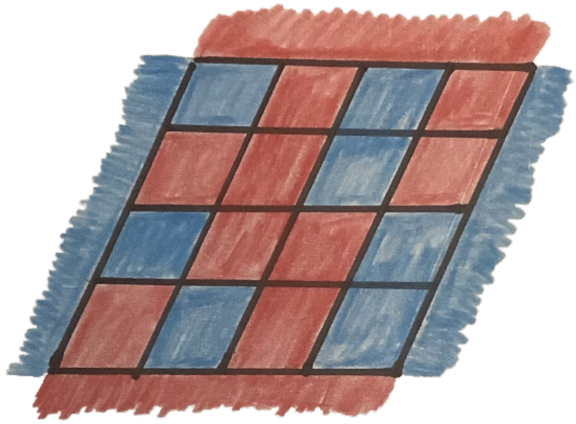

FIGURE 4. An example Hex gameboard

Proof. We start by abstracting the problem. This means converting our game board into a graph where colorable vertices are the cells and the edges represent the adjacency of cells. Note that the coloring discussed here does not have the restriction that adjacent vertices have different colors. Now our goal is to prove that for any coloring of the graph there always exists a path of vertices all colored c that connect opposite sides of color c.

Now we assume a contradiction that there is a draw. Now we define four sets:

$$R_0 = \{\text{Red vertices that are in the a red path starting from the bottom}\}$$
$$R_1 = \{\text{Red vertices not in } R_0\}$$
$$B_0 = \{\text{Blue vertices that are in the a blue path starting from the bottom}\}$$
$$B_1 = \{\text{Blue vertices not in } B_0\}$$

Now let e_1 be a rightward shift of a vertex parallel to the top and bottom edges of the gamboard and let e_2 be an upward shift of a vertex parallel to the left and right edges of the gameboard. Now we define a function f that maps vertices to vertices

$$f = \begin{cases} v + e_2 & v \in R_0 \\ v - e_2 & v \in R_1 \\ v + e_1 & v \in B_0 \\ v - e_1 & v \in B_1 \end{cases}$$

Since we have a draw, this function would not shift vertices off the board so everything is defined. Now let us consider triangles in the graph. Each point in a triangle that have vertices v_1, v_2, and v_3 can be expressed as x $= \sum x_i v_i$ where $\sum x_i = 1$. Now we apply our function on our triangles to get $f(x) = \sum x_i f(v_i)$ whichs is continuous. Now we get a homomorphism between G and the disc D^2 so $f : D^2 \to D^2$. This means we can apply Brouwer's Fixed Point theorem and find that f must have a fixed point. Now let $\epsilon_i = \{\pm e_1, \pm e_2\}$ so $f(v_i) = v_i + \epsilon_i$. This means that

$$\sum x_i f(v_i) = \sum x_i v_i \implies \sum x_i v_i + \sum x_i \epsilon_i = \sum x_i v_i \implies \sum x_i \epsilon_i = 0.$$

Without loss of generality, we can let $x_1 > 0$ and $\epsilon_1 = e_1$. Now one of the other epsilons must be $-e_1$. This results in two vertices of the same triangle being split among R_0, R_1, B_0, and B_1 which is a contradiction. ■

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