INTRODUCTION TO HYPERPLANE ARRANGEMENTS

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Abstract. In this paper, we will look at some important topics, theorems, etc. relating to Hyperplane Arrangements. We begin by introducing the basic idea in terms of a set of hyperplanes in a vector space. Once we have the basics, we can look at arrangements such as the braid arrangement, Shi arrangement, lineal arrangement, Catalan arrangment, and various others. The majority of this paper is based on Richard Stanley's "An Introduction to Hyperplane Arrangements" [Sta06]

1. Introduction to hyperplanes

First, let's get a general sense of what a hyperplane is. Generally speaking, a *hyperplane* of an n-dimensional space V is a subspace of dimension $n-1$. Hyperplanes may be of various spaces, such as affine, vector, or projective space, which we will define shortly. We may visualize one possible example of a hyperplane in Figure 1.

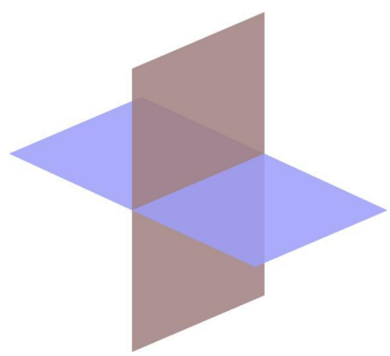

Figure 1. A plane is a 2-D hyperplane when embedded in a 3-D space.

In the following definition, we will cover the basic miscellaneous information as well as the vector/affine space. However, in our later study of the projective space, I will assume the reader's knowledge of vector spaces for definitions.

Background 1.1. A *transformation* is an invertible mapping from a set to itself but retains some geometric aspect. A *group homomorphism* from $(G, *)$ to $(H, \cdot)$ is a function $f : G \to H$ such that for all $a, b \in G$, we have that $f(a * b) = f(a) \cdot f(b)$, where $*$ and $\cdot$ are some operations. A *group action* on a space S is group homomorphism of some group into the group of transformations of S. We call the group with addition as its group operation the *additive group*. Finally, recall that a *vector space* is the set of vectors. Putting this all together, we have the definition of an *affine space*: it is a set A together with a vector space $\overrightarrow{A}$, and a transitive and free action of the additive group of $\overrightarrow{A}$ on the set A.

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Now we shall give a more formal definition of the hyperplane:

Definition 1.2. A *linear hyperplane* is an $(n-1)$-dimensional subspace of V,

$$H = \{v \in V : \alpha \cdot v = 0\},$$

where α is a fixed nonzero vector in V and $\alpha \cdot v$ is the dot product

$$(\alpha_1, \dots, \alpha_n) \cdot (v_1 \dots, v_n) = \sum \alpha_i v_i.$$

An *affine hyperplane* is a translation

$$J = \{v \in V : \alpha \cdot v = a\}$$

of a linear hyperplane, where α is as defined previously and $a \in K$.

Now that we have some background information on hyperplanes, we can now define a hyperplane arrangement.

Definition 1.3. A *finite hyperplane arrangement* $\mathcal{A}$ is a finite set of affine hyperplanes in vector space $V \cong K^n$, where K is some field.

Remark 1.4. Typically, we take $K = \mathbb{R}$, and this is assumed throughout the paper unless otherwise noted. Also, it is worth pointing out that in addition to finite hyperplane arrangements, these can also be infinite. However, we will not be discussing the infinite case, so from here on we will refer to finite hyperplane arrangements as simply *arrangements*.

Let's continue going through some of the basic definitions, as they will set the basis for further exploration. First, let us define a *linear form* as a linear map from a vector space to its field of scalars.

Definition 1.5. Suppose the equations of the hyperplanes of $\mathcal{A}$ are given by $L_1(x) = a_1, \dots, L_m(x) = a_m$, where $x = (x_1, \dots, x_n)$ and each $L_i(x)$ is a homogeneous linear form. Then,

$$\mathcal{Q}_{\mathcal{A}}(x) = (L_1(x) - a_1) \cdots (L_m(x) - a_m)$$

is called the *defining polynomial* of $\mathcal{A}$.

We may use the defining polynomial to describe the arrangement.

Example. If we fix a single coordinate of an n-dimensional coordinate space, then we obtain a *coordinate hyperplane*, which is an $(n-1)$-dimensional space. If we have an arrangement $\mathcal{A}$, which consists of n coordinate hyperplanes , then we have $\mathcal{Q}_{\mathcal{A}}(x) = x_1 x_2 \cdots x_n$.

Next, let's define dimension and rank.

Definition 1.6. The *dimension* $\dim(\mathcal{A})$ of $\mathcal{A}$ is $\dim(V) = n$, and the *rank* $\operatorname{rank}(\mathcal{A})$ of $\mathcal{A}$ is dimension of the space spanned by the normals (objects perpendicular) to the hyperplanes in $\mathcal{A}$. Furthermore, we say that $\mathcal{A}$ is *essential* if $\operatorname{rank}(\mathcal{A}) = \dim(\mathcal{A})$.

After what we've covered so far, we have a corollary (from Richard Stanley's paper [Sta06]) involving a bit of linear algebra.

Corollary 1.7. *Let*

$$W = \{v \in V : v \cdot y = 0 \textit{ for all } y \in Y\},$$

and let H be some element of $\mathcal{A}$. We have that $H \cap W$ is a hyperplane of W, so the set $\mathcal{A}_W := \{H \cap W : H \in \mathcal{A}\}$ is an essential arrangement in W.

Proof. Suppose that $\mathrm{rank}(\mathcal{A}) = r$ and $V = K^n$. Define Y as a complementary space in K^n to the subspace X, which is spanned by normals to hyperplanes in $\mathcal{A}$. If the characteristic[1] $\mathrm{char}(K) = 0$, then $W = X$. Furthermore, for every $H \in \mathcal{A}$, we have

$$\mathrm{codim}_W(H \cap W) = 1.$$

Here, $H \cap W$ is a hyperplane of W, so the set $\mathcal{A}_W := \{H \cap W : H \in \mathcal{A}\}$ is an essential arrangement in W. □

Now, let's look at a few more basic definitions:

Definition 1.8. A (closed) *half-space* is a set $\{x \in \mathbb{R}^n : x \cdot \alpha \geq c\}$, for some $\alpha \in \mathbb{R}^n$ and $c \in \mathbb{R}$. If H is a hyperplane in $\mathbb{R}^n$, then the complement $\mathbb{R}^n - H$ has two (open) components whose closures are half-spaces.

In other words, $\overline{R}$ is an n-dimensional convex polyhedron. Note that a *convex polyhedron* is the convex hull (it is the smallest convex set containing a shape) of finitely many points. A *bounded* polyhedron is called a (convex) *polytope*. Therefore, if either R or $\overline{R}$ is bounded, then $\overline{R}$ is an n-dimensional polytope.

Definition 1.9. An arrangement $\mathcal{A}$ is said to be in *general position* if

(1) $\{H_1, \ldots, H_p\} \subseteq \mathcal{A}, p \leq n \dim(H_1 \cap \cdots \cap H_p) = n - p$,

(2) $\{H_1, \ldots, H_p\} \subseteq \mathcal{A}, p > n H_1 \cap \cdots \cap H_p = \varnothing$.

Example. The set of two lines is in general position if two of them are not parallel and three of them do not meet at a point.

Now that we have finished seeing most of the basics, we can look at one example of a hyperplane arrangement.

Example. The *braid arrangement* $\mathcal{B}_n$ in K^n consists of the hyperplanes

$$\mathcal{B}_n : x_i - x_j = 0, 1 \leq i < j \leq n.$$

There are plenty of interesting things to be said about this arrangement, which has $\binom{n}{2}$ hyperplanes. First, let's look at Figure 2 for a visual representation of a possible braid arrangement. Note that for $K = \mathbb{R}$, a *region* of an arrangement $\mathcal{A}$ is a connected component of the complement X of the hyperplanes:

$$X = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H.$$

Let $\mathcal{R}(\mathcal{A})$ denote the set of regions of $\mathcal{A}$, and let

$$r(\mathcal{A}) = \#\mathcal{R}(\mathcal{A})$$

denote the number of regions. In the case of the braid arrangement in Figure 2, we have $r(\mathcal{A}) = 6$. Note that the closure of a region R of $\mathcal{A}$, denoted $\overline{R}$, is a finite intersection of half-spaces.

When counting the number of regions, we can consider whether $a_i < a_j$ or $a_i > a_j$, because this is the same as considering which side of the hyperplane $x_i - x_j = 0$ a given point

[1]The *characteristic* of a ring R, denoted $\mathrm{char}(R)$, is the least number of times summing the ring's multiplicative identity to obtain the additive identity.

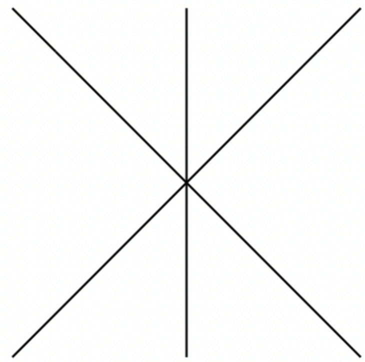

FIGURE 2. The braid arrangement, $\mathcal{A}_3$.

lies on. Note that for all permutations $w \in \mathfrak{S}_n$ (permutations of $\{1, 2, \ldots, n\}$), there is a corresponding region of $\mathcal{B}_n$, namely

$$R_w = \{(a_1, \ldots, a_n) \in_{\mathbb{R}^n} : a_{w(1)} > a_{w(2)} > \cdots > a_{w(n)}\}.$$

This gives us a surprisingly simple result: $r(\mathcal{B}_n) = n!$. This is an interesting case, because the calculation of number of regions is typically not this simple. The braid arrangement is not essential, because $\operatorname{rank}(\mathcal{B}_n) = n - 1$. If $\operatorname{char}(K) \nmid n$, then the space $W \subseteq K^n$ is

$$W = \{(a_1, \ldots, a_n) \in K^n : a_1 + \cdots + a_n = 0\}.$$

From the braid arrangement $\mathcal{B}_n$, we have similar arrangements, namely, the generic/semigeneric braid arrangements, Shi arrangement, linial arrangement, Catalan arrangement, semiorder arrangement, and threshold arrangement.

Example. The *generic braid arrangement* is given by $x_i - x_j = a_{ij}$, where each a_{ij} is generic, meaning that linearly independent over the prime field. The *prime field* of K is the smallest subfield which is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ if p is some prime. Similarly, the *semigeneric braid arrangement* is given by $x_i - x_j = a_i$, where each a_i is generic.

Example. The *Shi arrangement* is given by $x_i - x_j = 0, 1$, which implies the total number of hyperplanes is $n(n - 1)$.

Example. The *semiorder arrangement* is given by $x_i - x_j = -1, 1$.

Example. The *threshold arrangement* is given by $x_i - x_j = 0$. This example is not actually a form of the braid arrangement, but it is similar.

Example. The *linial arrangement* is given by $x_i - x_j = 1$.

Example. The *Catalan arrangement* is given by $x_i - x_j = -1, 0, 1$.

Although the following are beyond the scope of this paper (a basic introduction to hyperplane arrangements), the reader may enjoy learning more about hyperplane arrangements in projective spaces and how arrangements relate to the characteristic polynomial.

References

[Sta06] Richard P Stanley. An introduction to hyperplane arrangements, Feb 2006.