LINEAR PROGRAMMING

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ABSTRACT. A mathematical way of optimizing a function is called Linear Programming. Linear Programming allows researchers to find the most efficient solution to a problem given a set of limitations. In this paper, I will be taking you through a sample problem on optimization using the simplex algorithm and then I explore the algorithm on a general problem.

1. Introduction and Notation

Broadly speaking, linear programming is a method used to optimize a function subject to some number of constraints. Generally, the number of constraints defines the number of intersection points as well as the number of lines used in the graphical interpretation of the problem.

There are a few basic requirements for a problem to be a linear function:

- As the name suggests, all of the equations that serve as the constraints must be linear. This means that there are no variables multiplied together (e.g. xy).
- All values must be real.
- There are lower bounds of zero on all variables if none other are specified.
- The general form is:
  - indecision variables x_i
  - Constraints are either $\leq$ or $\geq$ symbols (Note that $<$ is congruent to $\leq$ and $>$ is congruent to $\geq$ because all values are real.
  - Variables are generally non-negative (non-negativity constraint)
  - Objective is to minimize or maximize some function

$$z = \sum_{i=0}^{n} c_i x_i$$

$$\text{subject to } a_{j1}x_1 + a_{j2}x_2 ... a_{jn}x_n \begin{cases} \leq b_j \\ \geq b_j \\ = b_j \end{cases} \text{ and } x_k = \leq 0$$

When a linear programming problem is in standard form, there are certain tighter restrictions: the constraints must be in the form of equalities (except for the non-negativity constraint), and all variables must be restricted to non negative. There is also a completely different form the problem is written in. Given the constant

Date: July 2021.

matrices A, B, and C^T, and the variable x, our new problem is of the form

$$\{\text{min or max}\}\ C^T x$$
$$\text{subject to } Ax = B, x \geq 0$$

2. The Process, Examples, and Introducing the Problem

To solve a linear programming question, we would first find a formulation of the problem, and then proceed to use an algorithm to evaluate the formulation. Suppose we have the following question:
Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes 40 USD an hour at Job I, and 30 USD an hour at Job II, how many hours should she work per week at each job to maximize her income?

Like we had earlier stated, we must find a formulation. To do this, we will start by defining variables.
Let x_1 be the number of hours Niki spends at her first job per week and let x_2 be the number of hours Niki spends on her second job per week.
Since we know that her first job gives her 40 USD and her second job gives her 30 USD, and we are trying to maximize the money Niki makes, we are trying to maximize the function

$$40x_1 + 30x_2$$

Since we know that Niki works less than 12 hours a week, we know that the sum of the hours worked in job 1 and the hours worked in job 2 is less than 12, so we have

$$x_1 + x_2 \leq 12$$

We also know that the preparation time for her first job is 2 hours, whilst her second job's preparation time is one hour. We know that she spends a maximum of 16 hours per week on her job, so we have

$$2x_1 + x_2 \leq 16$$

Lastly, we have the non-negativity constraint: we know that Niki can never work negative hours in a week, so we have that

$$x_1 \geq 0$$
$$x_2 \geq 0$$

To convert an inequality to an equality, we can add a "slack" variable to even out both sides. For example,

$$x_1 + x_2 \leq 12 \text{ would become } x_1 + x_2 + y_1 = 12$$
$$2x_1 + x_2 \leq 16 \text{ would become } 2x_1 + x_2 + y_2 = 12$$

The objective function Z would change to $Z - 40x_1 - 30x_2 = 0$
We can make an augmented matrix:

$$\begin{bmatrix} x_1 & x_2 & y_1 & y_2 & Z & \text{constant} \\ 1 & 1 & 1 & 0 & 0 & |12 \\ 2 & 1 & 0 & 1 & 0 & |16 \\ -40 & -30 & 0 & 0 & 1 & |00 \end{bmatrix}$$

To get the basic solution, we can plug in $x_1 = 0, x_2 = 0$ arbitrarily to get $(y_1, y_2, Z) = (12, 16, 0)$ Next, we choose a column that will reduce the objective function the most. We can call this the "reducing column". We look for the least value in the objective function row in the matrix.
This will be the first column in our case.
Next, we will be calculating the quotients for each row. The smallest possible quotient obtained identifies a row. To find the quotient, we divide the entries in the far right column by the entries in column 1, excluding the bottom row.
Our quotient for row 1 is $\frac{12}{1} = 12$.
Likewise, our quotient for row 2 is $\frac{16}{2} = 8$.
The pivot element is defined by the intersection of the smallest quotient row and the "reducing column." In our case, the pivot element is the intersection of column 1 and row 2, which is a 2.
The simplex method begins at a vertex where all x's are 0. It then moves from this point, increasing the value of the objective function.The value of the objective function is increased when we change the number of units of a variable. This can be done by adding the number of units of one variable while discarding the units of another.
Now, we will obtain the row echelon form (see end of document) of this augmented matrix. If we divide the second row by 2, we get:

$$\begin{bmatrix} x_1 & x_2 & y_1 & y_2 & Z & \cdot \\ 1 & 1 & 1 & 0 & 0 & |12 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & |08 \\ -40 & -30 & 0 & 0 & 1 & |00 \end{bmatrix}$$

We wish to obtain a 0 both above and below the pivot element. To do this, we multiply the second row by -1 and add it to the first one. We will also multiply the second row by 40 and add it to the last row. We get our new matrix as:

$$\begin{bmatrix} x_1 & x_2 & y_1 & y_2 & Z & \cdot \\ 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & |004 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & |008 \\ 0 & -10 & 0 & 20 & 1 & |320 \end{bmatrix}$$

At this point, we check if there are any negative numbers in the last row. We see that there is, so we re-do the algorithm, starting from calculating the quotient.
Taking column 2, we find the quotient to the first row: $\frac{4}{\frac{1}{2}} = 8$.
Similarly, on row 2, we have $\frac{8}{\frac{1}{2}} = 16$.

We can see that 8 is the smaller quotient here, so we have that the new pivot element is the intersection between row 1 and column 2, which is $\frac{1}{2}$. To make the pivot element 1, we first multiply by 2 to get:

$$\begin{bmatrix} x_1 & x_2 & y_1 & y_2 & Z & \cdot \\ 0 & 1 & 2 & -1 & 0 & |008 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & |008 \\ 0 & -10 & 0 & 20 & 1 & |320 \end{bmatrix}$$

To make the the other entries in this column equal to 0, we multiply the first row by $\frac{1}{2}$ and subtract it from row 2. Next, we multiply row 1 by 10 and add it to row 3. This gives us:

$$\begin{bmatrix} x_1 & x_2 & y_1 & y_2 & Z & \cdot \\ 0 & 1 & 2 & -1 & 0 & |008 \\ 1 & 0 & -1 & 1 & 0 & |004 \\ 0 & 0 & 20 & 10 & 1 & |400 \end{bmatrix}$$

We then observe that there are no longer any negative entries in the final row, so we are finished.

To interpret this result, we can take a look at the bottom row. This row corresponds to the equation

$$0x_1 + 0x_2 + 20y_1 + 10y_2 + Z = 400$$

. Moving terms and manipulating, we have

$$400 - 20y_1 - 10y_2 = Z$$

We know that we are trying to maximize Z. We also know that all variables are non negative. This means that the maximum will occur when $y_1 = y_2 = 0$ and $Z = 400$. Since we know that $y_1 = y_2 = 0$, we can look at the columns 1,2,5, and 6, ignoring the y_1 and y_2 column.

This gives us the matrix:

$$\begin{bmatrix} x_1 & x_2 & Z & \cdot \\ 0 & 1 & 0 & |008 \\ 1 & 0 & 0 & |004 \\ 0 & 0 & 1 & |400 \end{bmatrix}$$

The matrix reads $x_1 = 4$, $x_2 = 8$, and $Z = 400$. This tells us that if Niki works 4 hours at Job 1 and 8 hours at Job 2, she will maximize her income at 400 USD per week.

3. The General Simplex Method

Let us set some variable m equal to the number of rows in the matrix A. Likewise, we can set some variable n equal to the number of x's. Typically, $m \leq n$.

The simplex method assumes we have a solution in the feasible region x_{initial} that has exactly m of the x's as nonzero.

The mx values is equal to the multiplication of m columns of A, which we can refer to as the basis.

This set of basis vectors is linearly independent.The essence of the simplex function is to exchange one of the columns in the basis with another column outside the basis, effectively increasing one of the nonbasic variables while the constraints are still satisfied, reducing some of the x's.

We will increase the x that reduces the objective function the most, until one of the basic variables is 0. This changes our basis. Additionally, the solution is a new vertex of the simplex that is defined by the constraints. This move also exchanges one vertex of the simplex for one of its neighbors and moves the objective function towards the minimum.

References

[1] Josh C Nash *The (Dantzig) Simplex Method for Linear Programming* [*Algorithms*]. 1521-9615/00, 2000.
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[3] Christelle Guéret, Christian Prins, Marc Sevaux *Applications of optimization with Xpress-MP* `https://www2.isye.gatech.edu/ sahmed/isye3133b/XpressBook.pdf`
[4] Rupinder Sekhon and Roberta Bloom *Maximization By The Simplex Method*
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4. Footnotes

A well-known simplified form of a matrix is called the row echelon form.
A matrix is in row echelon form when the following constraints are satisfied:

- The first non-zero element in each row, called the leading entry, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.

Little of this content is original, most is research off of websites and articles you see in the list above. I have reworked through these problems, so the material might differ slightly, but the general idea remains unchanged.