Contents

Abstract

The concept of a matroid was first introduced by Hassler Whitney to bridge linear algebra and graph theory. Shortly after Whitney's publication, it was noted that matroids were also useful in projective geometry, transversal theory, amongst other areas. This paper will serve as an introduction to finite matroid theory, introducing some important terminology - such as rank, duality, deletion, and contraction - while simultaneously highlighting some fascinating properties about matroids. In particular, we will discuss an algorithm to find basis of maximal weight and prove the matroid union theorem. Along the way, we will also prove cryptomorphic definitions for some terms, like bases and rank.

1 Introduction to Matroid Theory

1.1 Matroid Definition

Definition 1. A matroid is a set E, often called the ground set, coupled with a collection of independent subsets of E, denoted $\mathcal{I}$ subject to the following:

(I1) $\emptyset \in \mathcal{I}$.

(I2) All subsets of an independent set are independent.

(I3) If $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, then there exists an $e \in I_2 \setminus I_1$ with $I_1 \cup \{e\} \in \mathcal{I}$.

Example 1. Perhaps the simplest example of a matroid is a free matroid, where $\mathcal{I}$ consists of all subsets of E. We can verify this is a matroid; (I1) holds because the empty set is a subset of E; (I2) holds because any subset of a subset of E is still a subset of E; a stronger rendition of (I3) holds because for all $e \in I_2 \setminus I_1$, $I_1 \cup \{e\} \in \mathcal{I}$.

Example 2. Another simple example of a matroid is the uniform matroid, denoted $U_{n,k}$. If $|E| = n$ and we let $\mathcal{I}$ consist of all subsets of E with cardinality less than or equal to k, then $(E, \mathcal{I})$ represents $U_{n,k}$ and indeed forms a matroid. We can verify this: (I1) holds if $k \geq 0$, (I2) holds by construction, and the stronger rendition of (I3) holds, since for all $e \in I_2 \setminus I_1$, $I_1 \cup \{e\} \in \mathcal{I}$. Notice that the free matroid is just the uniform matroid $U_{n,0}$.

Example 3. Matroids are also deeply connected to graphs, and we can form a matroid out of a graph. We can let E represent the collection of edges and let some subset of edges be independent iff it is acyclic (i.e. a spanning forest). We can verify that such a construction forms a matroid. (I1) and (I2) hold because with no edges, we cannot form a cycle, and any subgraph of an acyclic graph is bound to be acyclic as well. As for (I3), if there exists an edge $e = (u, v)$ where $v \in I_2 \setminus I_1$, then $I_1 \cup \{e\}$ is bound to stay acyclic. Hence, if V_j represents the set of nodes in I_j, then $V_2 \subseteq V_1 \Longrightarrow |V_1| > |V_2|$. By nature of forests if I_1 and I_2 have C_1 and C_2 connect components respectively, then:

$$|V_1| = |I_1| + C_1 \geq |I_2| = |I_2| + C_2.$$

Since $|I_1| < |I_2|$, $C_1 > C_2$, so there must be an edge in $I_2 \setminus I_1$ that bridges to connected components of I_1 together. As such, (I3) will always hold.
We denote such a matroid constructed from a graph G as $M(G)$.

Example 4. We can also form matroids out of vector spaces. Suppose we have a vector space V: we can let the ground set E be the set of objects in V and $\mathcal{I}$ be the collection of linearly independent subsets of E. As for verifying that $(E, \mathcal{I})$ forms a matroid, (I1) and (I2) clearly hold. Moving on to (I3), assume that it did not hold or that there was an $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, for all $e \in I_2 \setminus i_1$, $I_1 \cup \{e\} \notin \mathcal{I}$. This means that:

$$I_2 \setminus I_1 \subseteq \text{span}(I_1) \Longrightarrow I_2 \subseteq \text{span}(I_1) \Longrightarrow \text{span}(I_2) \subseteq \text{span}(I_1).$$

Since I_1 and I_2 are independent, this means that $|I_2| \leq |I_1|$, a contradiction to the assumption that $|I_1| < |I_2|$. Hence, indeed, (I3) holds and $(E, \mathcal{I})$ forms a matroid.

1.2 Bases

Like bases in vector spaces, we also have bases of matroids:

Definition 2. The basis of a matroid is a maximal independent set.

Lemma 1. The bases in a matroid all have the same cardinality.

Proof. Assume, by sake of contradiction, that we have two bases B_1, B_2, where, without loss of generality, $|B_1| > |B_2|$. In such a case, by (I3) we can chose an $e \in B_2 \setminus B_1$ such that $B_1 \cup \{e\} \in \mathcal{I}$. However, this contradicts the alleged maximality of B_1, and so we reach a contradiction. It follows that all bases have the same cardinality. ■

Lemma 2. The set of bases, $\mathcal{B}$, satisfies the following:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) If $B_1, B_2 \in \mathcal{B}$, then given an $x \in B_1 \setminus B_2$, there is an $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Proof. (B1) is clearly true, since we are examining finite matroids, so we devote most of our attention to (B2). By (I2), we know that given an $x \in B_1 \setminus B_2$, $B_1 \setminus \{x\}$ is independent, and since all bases have the same cardinality, $|B_1 \setminus \{x\}| < |B_2|$. Therefore, by (I3), there must exist a $y \in B_2 \setminus (B_1 \setminus \{x\}) = B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{I}$. However, notice that $(B_1 \setminus \{x\}) \cup \{y\}$ has the same cardinality as B_1, so it must be a basis as well. And so, the desired follows. ■

Interestingly enough, (B1) and (B2) characterize bases:

Lemma 3. If E is some arbitrary set, and $\mathcal{B}$ is a set obeying (B1) and (B2), with $\mathcal{I}$ the collection of subsets of elements of $\mathcal{B}$, then $(E, \mathcal{I})$ forms a matroid.

Proof. It is not hard to see that (I1) and (I2) hold, so we will devote our attention to showing that (I3) holds too. Suppose, by sake of contradiction, that (I3) did not hold for some independent sets I_1, I_2 with $|I_1| < |I_2|$. Since $I_1, I_2 \in \mathcal{I}$, they must be contained in some elements of $\mathcal{B}$, say B_1 and B_2. Notice that $(I_2 \cap B_1) \setminus I_1 = \emptyset$. If, on the contrary, there existed some element $e \in (I_2 \cap B_1) \setminus I_1$, then $I_1 \cup \{e\}$ would be independent, contradicting the assumption that (I3) fails. As a consequence of $(I_2 \cap B_1) \setminus I_1$ being empty, we have that:

$$I_2 \setminus I_1 = I_2 \setminus B_1.$$

As it is not clear how to proceed, we can artificially create constraints on B_1 and B_2. In particular, there exist a valid B_1, B_2 such that $B_2 \setminus (I_2 \cup B_1) = \emptyset$. As such, we can constrain B_1, B_2 to minimize $|B_2 \setminus (I_2 \cup B_1)|$. With this constraint, it is clear why $B_2 \setminus (I_2 \cup B_1) = \emptyset$; if there was an element $e \in B_2 \setminus (I_2 \cup B_1)$, then there would also be an $f \in B_1 \setminus B_2$ such that $(B_2 \cup \{f\}) \setminus \{e\} \in \mathcal{I}$. However, $(B_2 \cup \{f\}) \setminus \{e\}$ would be a better choice than B_2. So, indeed, $B_2 \setminus (I_2 \cup B_1) = \emptyset$ and as such:

$$B_2 \setminus B_1 = I_2 \setminus B_1.$$

Seeing the intrinsic symmetry between B_1 and B_2, we may also wonder if $B_1 \setminus (I_1 \cup B_2)$ is empty. Indeed, it is, and we can see so by assuming the contrary. If there existed some $x \in B_1 \setminus (I_1 \cup B_2)$, then we could always find a corresponding $y \in B_2 \setminus B_1 = I_2 \setminus I_1$ with $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. But this contradicts our assumption that (I3) fails for I_1, I_2, since $I_1 \cup \{y\} \subseteq (B_1 \setminus \{x\}) \cup \{y\}$. Thus, we have that:

$$I_1 \setminus B_2 = B_1 \setminus B_2 \subseteq I_1 \setminus I_2.$$

However, this implies that $|I_2 \setminus I_1| = |B_2 \setminus B_1| = |B_1 \setminus B_2| \leq |I_1 \setminus I_2|$, which contradicts the assumption that $|I_1| < |I_2|$. As such, we reach a contradiction, and the desired follows. ■

Proposition 1. For some graph G, $M(G)$'s bases are the spanning trees of G.

1.3 Bases of Maximal Weight

Suppose that we have a matroid and a weight function that assigns elements of E positive weights, and we want to chose the basis of maximal weight. How would we do that. We claim that the following algorithm, which we call algorithm 1, returns a basis of maximal weight:

- Initialize a set $S = \emptyset$; after the end of this algorithm, S will be the basis of maximal weight.
- Iterate through the elements in E in decreasing order of weights, and add an element $e \in E$ if $S \cup \{e\} \in \mathcal{I}$.

Lemma 4. Algorithm 1 produces a basis of maximal weight.

Proof. We alternatively prove that at each iteration, S can be extended to a basis of maximal weight. Let the status of S at the ith iteration be S_i, indexed such that $S_0 = \emptyset$. We claim that if S_i can be extended to a basis B_i of maximal weight, then S_{i+1} can also be extended to a basis of maximal weight. Indeed, this is clearly true if $S_i = S_{i+1}$ or in other words, we do not pick the $i+1$th element. Similarly, if the $i+1$th element belongs to S_i, then we can let $B_{i+1} = B_i$.

Let us call the $i+1$th element we consider e_{i+1}. We only need to consider what happens when $S_i \cup \{e_{i+1}\} \in \mathcal{I}$ and $e_{i+1} \notin B_i$. In such a case, by (B2), $S_i \cup \{e_{i+1}\}$ can be extended to some arbitrary basis B', where $|B' \setminus B_i| = |B_i \setminus B'| = 1$ or in other words, $B' = (B_{i-1} \cup \{e_{i+1}\}) \setminus e_j$ for some $j > i+1$. Notice that the weight of B', denoted $w(B')$ is as follows:

$$w(B') = w(B_{i-1}) + w(e_{i+1}) - w(e_j) \geq w(B_i). \tag{1}$$

That is, B' has a weight that is equal to that of B_i. So indeed, we can let $B_{i+1} = B'$. That is, all S_i can be extended to a basis.

But the proof is not complete: it is possible that algorithm 1 does not actually return a basis. Suppose that this is the case. Then, we know that no element in $E \setminus S$ can be added to S while maintaining S's independence. That is, S cannot be extended to a basis. But this is a contradiction, so as desired, algorithm 1 does not only return a basis, but a basis of maximal weight. ∎

Note 1. We can slightly modify algorithm 1 to produce a basis of minimal weight. Simply iterate through the elements of E in increasing order of weights. The proof that algorithm 1 works is completely analogous.

Lemma 5. If for all weight functions w, algorithm 1 returns a basis of maximal weight, then $(E, \mathcal{I})$ is a matroid.

Proof. This is equivalent to showing that if $(E, \mathcal{I})$ is not a matroid, then we can find some weight function that makes algorithm 1 fails. There can be three reasons why $(E, \mathcal{I})$ is not a matroid: either (I1), (I2), or (I3) does not hold.

Suppose that (I2) does not hold for two sets I_1, I_2; that is, $I_1 \subset I_2$, $I_2 \in \mathcal{I}$, but $I_1 \notin \mathcal{I}$. Then, we want to construct a weight function that fails. Intuitively, if we want the algorithm to fail, we should first process elements of I_1, then the elements of I_2, and then the elements in neither I_1, nor I_2. So naturally, we construct the following weight function:

$$w(e_i) = \begin{cases} 2 & e_i \in I_1 \\ 1 & e_i \in I_2 \setminus I_1 \\ 0 & e_i \notin I_2 \end{cases}.$$

Hopefully, our weight function fails. Indeed, it will, since the final weighting of our basis is less than $2|I_1| + |I_2 \setminus I_1|$. However, this weighting is not optimal: we could have chosen the basis I_2 and achieved better weighting.

Now, consider the case where (I3) fails; that is, there is an $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ and for all $e \in I_2 \setminus I_1$, $I_2 \cup \{e\} \notin \mathcal{I}$. Intuitively, we want to ensure that we do not pick elements of $I_2 \setminus I_1$, so we come up with the following weight function:

$$w(e_i) = \begin{cases} 1 + \frac{1}{2|I_1|} & e_i \in I_1 \\ 1 & e_i \in I_2 \setminus I_1 \\ 0 & e_i \notin (I_2 \cup I_1) \end{cases}.$$

Algorithm 1 will produce a basis of weight:

$$|I_1| \cdot \left(1 + \frac{1}{2|I_1|}\right) = |I_1| + \frac{1}{2},$$

but we could have done better had we selected I_2: we would have been able to achieve at least $|I_2|$. And thus, algorithm 1 will fail in such a case. ∎

Question 1. Devise an algorithm that returns a maximal spanning forest of a graph.

Using algorithm 1, we can almost immediately devise such an algorithm:

- Initialize a set $S = \emptyset$; at the end of this algorithm, S will be the basis of maximal weight.
- Sort the edges $e \in E$ in decreasing order of weight.
- Add the edge e to S is $\{e\} \cup S$ is acyclic.

This is called Kruskal's algorithm. In practice, it is frequently used in sparse graphs (an alternative maximal spanning tree algorithm called Prim's algorithm is faster for dense graphs).

Question 2. Suppose we have a set of tasks to complete, each of which has a corresponding due date. Each task takes a day complete, and you cannot be on working at multiple tasks at once. For each task, if you complete it, you gain some points (the amount of points varies depending on the task). In what order should you attempt the tasks so as to maximize the amount of points received?

We need to transform this problem into the language of matroids. We need some notion of independence.

We can let any set of tasks which can all be completed before the deadline be independent. In particular, iff an element $S \in \mathcal{I}$, then all tasks in S can all be completed in time. We speculate that $(E, \mathcal{I})$ forms a matroid.

Indeed, $(E, \mathcal{I})$ does form a matroid. We can verify (I1): $\emptyset \in \mathcal{I}$ because naturally, if we chose zero tasks, we can complete them all in time. Similarly, (I2) holds: if we can complete some set of tasks, then we can also complete a subset of those tasks.

Proving (I3) is slightly more involved. Chose two sets X, Y that are independent with $|X| < |Y|$. Let t_0 be the first time in which more tasks in X have deadline before or at t_0 than in Y. That is, t_0 is the first point in time at which $N_t(X) \geq N_t(Y)$, where $N_t(A)$ is the number of tasks in A whose deadline is t or earlier. This definition of $N_t(A)$ is particularly telling because if for all $t \in [0, n]$, $N_t(A) \leq t$, then A is independent; so if we can find an $e \in Y \setminus X$ such that $N_t(X \cup \{e\}) \leq t$ for all $t \in [0, n]$, then we are done.

In attempt to find such an element e, let y be an element in $Y \setminus X$ such that its deadline is time $t_0 + 1$. We know that $N_t(X \cup \{y\}) = N_t(X) \leq t$ for all $t \in [0, t_0]$. At the same time, $N_t(X \cup \{y\}) = N_t(X) + 1 \leq N_t(Y) \leq t$ for all $t \in [t_0 + 1, n]$. That is, $N_t(X \cup \{y\}) \leq t$ for all t, so $X \cup \{y\}$ must be independent. Indeed, from this construction, (I3) holds and $(E, \mathcal{I})$ forms matroid.

Now that (I3) forms a matroid, we can modify algorithm 1 to find the optimal ordering of tasks:

- Initialize a set $S = \emptyset$.
- Sort R, the set of monetary rewards for completing a task, in decreasing order. Permute D, the set of deadlines, accordingly.
- Iterate over the tasks, in the order which the appear in R, and add a task t iff we can complete all tasks in $S \cup \{t\}$ before the deadline.

At the end of this process, by algorithm 1, we know that S will return a basis of maximal length. From here, formally solving (2) is left as an exercise to the reader.[1]

[1] As a challenge, devise an algorithm that works in $O(N \log N)$ time, where N is the number of tasks. This will require some other data structures, though.

1.4 Rank

There are many equivalent definitions of rank. Perhaps the simplest one is that rank maps a subset S of E to the cardinality of the maximal independent subset of S.

Lemma 6. The rank function $r : 2^E \to \mathbb{Z}$ satisfies the following properties:

(R1) $r(X) \in [0, |X|]$.

(R2) If $X \subseteq Y$, then $r(X) \le r(Y)$.

(R3) For all, X, Y:

$$r(X \cup Y) + r(X \cap Y) \le r(X) + r(Y).$$

Proof. Let the maximal independent subset of $X \cap Y$ be I_1, and extend I_1 to I_2, a maximal independent set in $X \cup Y$. We can partition $I_2 = I_3 \cup I_4 \cup I_5$, where $I_3 \in X \setminus Y, I_4 \in Y \setminus X$, and $I_5 \in X \cap Y$. It is not hard to see that $r(X \cup Y) = |I_2| = |I_3| + |I_4| + |I_5|$. Furthermore, we know that since $I_1 \cup I_3 \subseteq I_2$, $r(X) \ge |I_1| + |I_3|$. Similarly, $r(Y) \ge |I_1| + |I_4|$. Thus, we have that:

$$r(X) + r(Y) \ge 2|I_1| + |I_3| + |I_4| = r(X \cup Y) + r(X \cap Y),$$

from which the desired follows. ■

Lemma 7. Let $r : 2^E \to \mathbb{N}$, where r satisfies (R1) through (R3). Then, if $r(X \cup \{y\}) = r(X)$ for all $y \in Y$, $r(X \cup Y) = r(X)$.

Proof. We can prove so via induction. Suppose that $Y \setminus X = \{y_1, y_2, \dots, y_{n+1}\}$, and assume that the desired is true when $|Y \setminus X| \le n$. Notice that:

$$\begin{aligned} r(X) + r(X) &= r(X \cup \{y_1, \dots y_n\}) + r(X \cup \{y_{n+1}\}) \\ &\ge r(X \cup \{y_1, \dots y_n, y_{n+1}\}) + r(X), \end{aligned}$$

from the submodularity of the rank function. As a consequence, we see that $r(X \cup (Y \setminus X)) = r(X)$ or $r(X \cup Y) = r(X)$. ■

Lemma 8. Let E be some arbitrary set and $r : 2^E \to \mathbb{N}$, which satisfies (R1), (R2), and (R3). Let $\mathcal{I}$ be the collection of subsets S of E which satisfy $r(S) = |S|$. Then, $(E, \mathcal{I})$ forms a matroid.

Proof. Showing that (I1) holds is not terribly hard, so we devote our attention to proving (I2) and (I3). To prove (I2), let $I_1 \in \mathcal{I}$ with $I_2 \subseteq I_1$. By the submodularity of the rank function, we know that:

$$r(I_2 \cup (I_1 \setminus I_2)) + r(I_2 \cap (I_1 \setminus I_2)) \le r(I_2) + r(I_1 \setminus I_2),$$

which implies that:

$$|I_1| = r(I_1) \le r(I_2) + r(I_1 \setminus I_2) \le |I_2| + |I_1 \setminus I_2| = |I_1|.$$

The inequality must be equality and so $r(I_2) = |I_2|$, which implies that $I_2 \in \mathcal{I}$.
As for (I3), assume, by sake of contradiction, that it does not hold. Then, there must be an $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ for which $r(I_1 \cup \{e\}) \ne |I_1 \cup \{e\}|$ for all $e \in I_2 \setminus I_1$. Notice $|I_1| + 1 > r(I_1 \cup \{e\}) \ge r(I_1) = |I_1|$, so $r(I_1 \cup \{e\}) = |I_1|$. By the previous lemma, this implies that $r(I_1) = r(I_1 \cup I_2)$. However, this implies that $I_1 \ge I_2$, a contradiction. Thus, our original assumption that (I3) fails is fallacious. ■

Definition 3. An element for which $r(\{e\}) = 0$ is called a loop.

1.5 Duality, Deletion, and Contraction

Proposition 2. If M is a matroid with set of bases $\mathcal{B}$ and we define $\mathcal{B}^*(M)$ as $\{E \setminus B : B \in \mathcal{B}\}$, then $\mathcal{B}^*$ is the set of bases of some matroid M^*.

Proposition 3. M^* is a matroid with rank function:

$$r^*(X) = |X| - r(E) + r(E \setminus X).$$

Proof. (R1) and (R2) are clearly true, so we proceed to show that (R3) holds:

$$\begin{aligned} r^*(X \cap Y) + r^*(X \cup Y) &= |X \cup Y| + |X \cap Y| - 2r(E) + r(E \setminus X) + r(E \setminus Y) \\ &= |X| + |Y| - 2r(E) + r((E \setminus X) \cap (E \setminus Y)) + r((E \setminus X) \cup r(E \setminus Y)) \\ &\leq |X| + |Y| - 2r(E) + r(E \setminus X) + r(E \setminus Y) \\ &= r^*(X) + r^*(Y). \end{aligned}$$

Hence, indeed r^* is the rank function of some matroid, we just need to show that it is the rank function of the dual matroid. Given the rank function, we can construct the collection of independent sets by letting $\mathcal{I}^*$ be the collection of subsets of E for which their rank is equal to their cardinality. So, if I is a base of M^*, then

$$r^*(I) = |I| - r(E) + r(E \setminus I) = |I| = r^*(E).$$

That is, $r(E) = r(E \setminus I)$, so $E \setminus I$ is a basis in M. ∎

Definition 4. The deletion of S from M is defined to be:

$$M \setminus S = (E \setminus S, \{I \subseteq E \setminus S : I \in \mathcal{I}\}).$$

Definition 5. The restriction of M to S is defined to be:

$$M|_S = (S, \{I \subseteq S : I \in \mathcal{I}\}).$$

Definition 6. The contraction of S from M is defined to be:

$$M/S = (M^* \setminus S)^*.$$

Definition 7. A matroid M_1 is a minor of M if it can be achieved via a sequence of deletions and contractions.

Proposition 4. $r_{M/T}(S) = r_M(S \cup T) - r_M(T)$.

Proof. We know that:

$$\begin{aligned} r_{M/T}(S) &= |S| + r_{M^* \setminus T}(E \setminus (S \cup T)) - r_{M^* \setminus T}(E \setminus T) \\ &= |S| + r^*(E \setminus (S \cup T)) - r^*(E \setminus T) \\ &= |S| + |E \setminus (T \cup S)| - |E \setminus T| + r_M(T \cup S) - r_M(E) - r_M(T) + r_M(E) \\ &= |S| + |E \setminus (T \cup S)| - |E \setminus T| + r_M(T \cup S) - r_M(T) \\ &= r_M(T \cup S) - r_M(T). \end{aligned}$$

∎

Proposition 5. If G is a graph, then for all subsets T of G:

$$M(G) \setminus T = M(G \setminus T).$$

$$M(G)/T = M(G/T).$$

Proof. It is clear that $M(G) \setminus \{e\} = M(G \setminus e)$, so by induction, $M(G) \setminus T = M(G \setminus T)$. As for contractions, if e is a loop of G, then the desired holds, so we can assume it is not a loop. If I is acyclic in G/e, then $I \cup \{e\}$ must be acyclic in G. In other words, the set of independent sets in $M(G)/e$ is the same as the set of independent sets in $M(G/e)$, so indeed, by induction, the desired holds. ∎

Corollary 1. Graphic matroids are closed under taking minors.

Note 2. Analyzing excluded minors is a very rich topic in matroid theory. For example, based solely on forbidden minors, we can determine if a matroid is graphic. Similarly, just by looking at excluded minors, we can tell if a matriod is linear.

1.6 Matroid Union Theorem

Theorem 1 (Weighed Matroid Union Theorem)**.** If $M_1, M_2, \ldots M_n$ are matroids with rank fucntions $r_1, r_2, \ldots r_n$ operating on a common ground set E, then (1) and (2) are equivalent:

(1) There is a $w-$covering of E with $V_1, V_2, \ldots V_n$ and V_i independent in M_i.

(2) For all $A \subset E$:

$$\sum_{i=1}^{n} r_i(A) \geq \sum_{e \in A} w(e).$$

Proof. One direction is easy, namely showing that (1) $\Longrightarrow$ (2). In particular, we know that:

$$\sum_{e \in A} w(e) = \sum_{i=1}^{n} |A \cap V_i| \leq \sum_{i=1}^{n} r_i(A),$$

provided that there is a $w-$covering of E with $V_1, V_2, \ldots V_n$ and V_i independent in M_i.
As for the other direction, induct on E and suppose that equality for (2) holds for some A or in other words, that $\sum_{i=1}^{n} r_i(A) = \sum_{e \in A} w(e)$ for some A. We know that there is a $w|_A$-covering using $U_1, U_2, \ldots U_n$, where U_i is independent in $M_i|A$. Similarly, there is a $w|_{E \setminus A}$ covering of $E \setminus A$ on $M_1/A, M_2/A, \ldots M_n/A$ for some U_i'. We know that the new rank function $r_i'(S) = r_i(S \cup B) - r_i(S)$, so (2) is satisfied. Letting, $V_i = U_i \cup U_i'$, the desired follows.
However, it may not be the case that equality sometimes holds: we could have strict inequality. In such a case, pick an e for which $w(e) > 0$ and an i for which $r_i(e) > 0$. Consider $M_1, \ldots, M_i/e, \ldots M_n$ and a weight function w', where $w' \equiv w|_{E/e}$, but $w'(e) = w(e) - 1$. Since (2) holds, there is a covering using $V_1, \ldots, V_i', \ldots V_n$ by our inductive hypothesis. If we let $V_i = V_i' \cup \{e\}$, then V_is are a a valid $w-$covering. So indeed, the desired holds.
We have exhausted all cases, so indeed (1)$\Longleftrightarrow$(2). ∎

Corollary 2. if $M_1, M_2, \ldots M_n$ are matroids with rank functions $r_1, \ldots r_n$ operating on a common ground set E, then (1) and (2) are equivalent:

(1) There are $\cup_{i=1}^{k} V_i = E$, where V_i is independent in M_i for all i.

(2) For all $A \subset E$,

$$\sum_{i=1}^{n} r_i(A) \geq |A|.$$