HOMOMESIC FUNCTIONS

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1. Introduction

In the field of Dynamical Algebraic Combinatorics we explore actions on sets of discrete combinatorial objects, many of which can be built up by small local changes. One of the key terms in this field is that of *homomesy*. The term describes this behavior attributed to special functions (in the field of Dynamical Algebraic Combinatorics more frequently referred to as a statistic): Given a group action on a set of combinatorial objects, a function on these objects is called *homomesic* if its average value is the same over all orbits.

This in more elementary terms means that given some action with an orbit of order n, a function f is homomesic if on every point in the state space, $\frac{1}{n}(f(x)+f(Tx)+\cdots+f(T^{n-1}x))$ is constant regardless of the x. There are some very elementary functions which satisfy this property.

In this paper we will start with rigorously defining homomesy. Then we will move on to some basic examples of functions exhibiting homomesy, prove that this is the case, and show the consequences. Lastly, we will prove some more general theoretical results about homomesy relating to group actions on sets.

2. Defining Homomesy

Definition 2.1. Given a set $\mathcal{S}$, an invertible map $\tau : \mathcal{S} \to \mathcal{S}$ such that each τ-orbit is finite, and a function (or "statistic")

$$f : \mathcal{S} \to \mathbb{F}$$

where $\mathbb{F}$ is a field (with characteristic 0, but this is not really relevant), we say that $(\mathcal{S}, \tau, f)$ exhibits homomesy if and only if there exists some constant $c \in \mathbb{F}$ such that for every τ-orbit, $\mathcal{O} \subset \mathcal{S}$

$$\frac{1}{|\mathcal{O}|}\sum_{x\in\mathcal{O}} f(x) = c.$$

Here we say that f is a *homomesic* function (with respect to τ and $\mathbb{F}$) and refer the f as c-mesic.

When $\mathcal{S}$ is a finite set, the homomesy of $f : \mathcal{S} \to \mathbb{K}$ can be restated in terms of global averages and say that all all orbit averages for a given transformation is equal to the global average of the statistic f, or symbolically written as

$$\frac{1}{|\mathcal{O}|}\sum_{x\in\mathcal{O}} f(x) = \frac{1}{|\mathcal{S}|}\sum_{x\in\mathcal{S}} f(x)..$$

Thus we get the triple $(\mathcal{S}, \tau, f)$ is homomesic if and only if for every t-orbit we have that

$$\frac{1}{|\mathcal{O}|}\sum_{x\in\mathcal{O}} f(x) = \frac{1}{|\mathcal{S}|}\sum_{x\in\mathcal{S}} f(x).$$

3. Homomesy in Binary Strings

Now, we turn to examples of homomesic functions.

Indicator functions of coordinates under cyclic rotation of binary strings. The following are some useful definitions we'll need for this section to study examples of homomesy.

(1) $S_{n,k}$ is the set of all binary strings with length n and exactly k $1's$.
(2) Given some $s \in S_{n,k}$ a cyclic rotation of s, denoted by C_R, is the new string in $s' \in S_{n,k}$ such that for each position i in s, $s'_{i+1 \mod n} = s_{i \mod n}$.
(3) An inversion of s, denoted

$$\text{Inv}(s) = |\{(i,j) : i < j, s_i = 1, s_j = 0\}|.$$

Now, let $\mathbb{1}_i(s) = s_i$, so this function returns 1 if $s_i = 1$.

Lemma 3.1. *The triple $(S_{n,k}, C_R, \mathbb{1}_i)$ exhibits homomesy and is k/n-mesic.*

Proof. For any i, $\mathbb{1}_i$ is k/n-mesic because in every superorbit, a 1 is in position i k different times and the average becomes k/n. So ■

Inversions under cyclic rotation of binary strings. We will need to look at the following terms.

Here, we should reiterate the necessary terms.

(1) $S_{n,k}$ is the set of all binary strings with length n and exactly k $1's$.
(2) Given some $s \in S_{n,k}$ a cyclic rotation of s, denoted by C_R, is the new string in $s' \in S_{n,k}$ such that for each position i in s, $s'_{i+1 \mod n} = s_{i \mod n}$.
(3) An inversion of s, denoted

$$\text{Inv}(s) = |\{(i,j) : i < j, s_i = 1, s_j = 0\}|.$$

Now, let's look at the behavior of inversions under the transformation of cyclic rotations.

Let $n = 4$ and $k = 2$. There are two orbits, generated by the following strings

$$s_1 = \langle 1,1,0,0\rangle, s_2 = \langle 1,0,1,0\rangle$$

and when we apply C_R to s_1 we get

$$C_R(s_1) = \{\langle 1,1,0,0,\rangle, \langle 0,1,1,0\rangle, \langle 0,0,1,1\rangle, \langle 1,0,0,1\rangle\}$$

and

$$C_R(S_2) = \{\langle 1,0,1,0\rangle, \langle 0,1,0,1\rangle\}.$$

For $C_R(s_1)$ the inversions come out to be

$$\text{Inv}(C_R(s_1)) = \langle 0,2,4,2\rangle$$

and

$$\mathrm{Inv}(C_R(s_2)) = \langle 1, 3\rangle.$$

Thus over each orbit, the average is 2.

Now, it is not necessary that the orbits be of the same size, just that the averages are the same for each orbit.

Lemma 3.2. *The triple* $(S_{n,2}, C_R, \mathrm{Inv})$ *exhibits homomesy and is* $n-2$*-mesic.*

We will not prove this lemma, but will go on to look at more general examples of this.

Now, we can see this more generally.

Theorem 3.3. *In general, if* $\mathcal{S} = S_{n,k}, \tau = C_R,$ *and* $f = \mathrm{Inv}$ *then* $(\mathcal{S}, \tau, f)$ *is* $\frac{k(n-k)}{2}$*-mesic.*

Proof. We're going to rephrase this problem by switching from 1 and 0 to 1 and -1 and let the number of 1's be a and the number of -1's be b.

Now, we prove this proposition to write the indicator function of (s_i, s_j) being an inversion pair as $\frac{1}{4}(1+s_i)(1-s_j)$. Then

$$f(s) = \sum_{i<j} \frac{(1+s_i)(1-s_j)}{4}$$

$$= \frac{1}{4}\sum_{i<j}(1 + s_i - s_j - s_i s_j)$$

$$= \frac{1}{4}\Big(\sum_{i<j} 1 + \sum_{i<j} s_i - \sum_{i<j} s_j - \sum_{i<j} s_i s_j\Big).$$

Then we get that

$$\sum_{i<j} s_i s_j = \Big(\frac{a(a-1)}{2} + \frac{b(b-1)}{2}\Big) - ab = \frac{n(n-1)}{2} - 2ab,$$

so

$$f(s) = \frac{1}{4}\Big(2a + \sum_{i<j} s_i - \sum_{i<j} s_j\Big) = \frac{ab}{2}.$$

Thus, we have that $(\mathcal{S}, \tau, f)$ is $\frac{k(n-k)}{2}$-mesic. ■

4. **Homomesy in Permutations**

Inversions in Permutations. Let S_n be the set of permutations of $[n] := \{1, 2, \ldots, n\}$. Now, define τ to be a map in S_n such that $s = (a_1 \cdots a_n)$ and $\tau(s) = s'$ where $s' = (a_n \cdots a_1)$.

Theorem 4.1. *We have that the number of inversions of a permutation is* $\frac{n(n-1)}{4}$*-mesic with respect to the permutation reversal.*

Proof. Now, let $f(\pi)$ be the number of inversions in π. Then, applying τ, because the order is two we get $f(\pi) + f(\tau(\pi)) = \frac{n(n-1)}{2}$, now dividing by 2 we get that f is homomesic under the action of τ, where $c = \frac{n(n-1)}{4}$. ■

Ballot Theorems. Fix two nonnegative integers a and b and let $n = a+b$. Then, let $\mathcal{S}$ be the set of strings of length n to get $(s_1, \ldots, s_n)$, consisting of a elements equal to -1 and b letters equal to 1. Now, let this be a two way election and we treat the a elements as votes for candidate A and b elements for candidate B. Now, if $a < b$ then B will be the winner and otherwise, A will be the winner. Now, there are $a+b$ ballots and let us ask for the probability that at every stage of the counting of the ballots, B be in the lead. This probability is the expected value of $f(s)$ where $f(s)$ is 1 if $s_1 + \cdots + s_i > 0$ for all $1 \leq i \leq n$ and is 0 otherwise, and where s is chosen uniformly at random from $\mathcal{S}$. Now, we have that this probability is exactly $\frac{b-a}{b+a}$.

Theorem 4.2. *Let $\tau := C_L : \mathcal{S} \to \mathcal{S}$ be the leftward cyclic shift operator that sends $(s_1, s_2, \ldots, s_n) \to (s_2, s_3, \ldots, s_n, s_1)$. Then over any orbit $\mathcal{O}$ one has*

$$\frac{1}{|\mathcal{O}|} \sum_{s \in \mathcal{O}} f(s) = \frac{b-a}{b+a}.$$

In other words, f is c-mesic with $c = \frac{b-a}{b+a}$.

5. **Linear Actions on Vector Spaces**

Let V be a vector space over a field $\mathbb{K}$ of characteristic 0, and define $f(v) = v$. Let $T : V \to V$ be a linear map of order n so that $T^n = I$. We say that v is invariant under T if $Tv = v$ and 0-mesic under T if

$$\frac{1}{n} \sum_{k=0}^{n} T^k v = 0.$$

Theorem 5.1. *Every v can be written uniquely as the sum of an invariant vector v and a 0-mesic vector w.*

Proof. Let's say we want to write ω in such a manner. Then

$$w = \frac{1}{n} \sum_{k=0}^{n} T^k \omega$$

and w would be either 0-mesic or invariant and then $\omega - w$ would be invariant if w was 0-mesic and 0-mesic if w was invariant and thus we'd have a sum of an invariant and a 0-mesic vector. ■

In representation-theoretic terms, we are applying symmetrization to v to extract from it the invariant component of v associated with the trivial representation of the cyclic group, and the homomesic $(0 - mesic)$ component of v consists of everything else.

Another way to think about this is in terms of the kernel of $(I - T)^n$.

Every element of the kernel of $(I-T)^n = (I-T)(I+T+T^2+\cdots+T^{n-1})$ can be written uniquely as the sum of an element of the kernel of $I-T$ and an element of the kernel of $I+T+T^2+\cdots+T^{n-1}$. This picture relates more directly to our earlier definition if we use the dual space V^* of linear functionals on V as the set of statistics on V.

Example. As a concrete example, let $V=\mathbb{R}^n$ and let T be the cyclic shift of coordinates sending

$$T:(x_1,x_2,...,x_n)\to(x_n,x_1,...,x_{n-1}).$$

The T invariant functionals form a 1 dimensional subspace of V^* spanned by the functional $(x_1,x_2,...,x_n)\to x_1+x_2+\ +x_n$, while the 0 mesic functionals form an $(n-1)$ dimensional subspace of V^* spanned by then — 1 functionals $(x_1,x_2,...,x_n)\to x_i-x_{i+1}$ (for $1\le i\le n-1$).

Example. Also, we can consider the ring $\mathbb{R}[x_1,...,x_n]$ of polynomial functions $p(x_1,x_2,...,x_n)$ on $\mathbb{R}^n$; this ring, viewed as a vector space over $\mathbb{R}$, can be written as the direct sum of the subspace of polynomials that are invariant under the action of T and the subspace of polynomials that are 0-mesic under the action of T.

6. **Bibliography**

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