Counting Permutations for Acceptable Harmonization in a Four-Voice Chorale

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The objective of this paper is to rigorously solve for the number of voice sequences that satisfy several objective rules of the Four-Voice chorale.

Although the technicalities of the Four-Voice chorale still follow the conventions of the 1600's, the meaning and use of this method have changed drastically over the years, from—at one point—being considered substantial enough to exist as independent piece of music, to an exercise for young composers, to a test bed for harmonic invention, to piano and choral etudes.

The method described is a system of rigorous rules for the arrangement or "voicing" of notes to create a predetermined harmony. Some of the finer details of these rules may vary depending on where you learn this method, and master composers such as Bach, Handel, and Schutz have such fine control over this method (and lived in a time where composition with this technique was so skillful it was its own art), that they bent the rules to create more appealingly melodic chorales. However, for the course of this paper I will examine only the most drastic and agreed-upon rules, with no concern for subjective musicality. The principles used will be

1. No parallel 5ths
2. No parallel 8ths
3. No unison
4. Voices shall not cross each other
5. The third and root of the harmony shall always be played
6. The 5th shall not be doubled
7. All voices must exist within their range.

The rules will be clarified notation with a combination of *Fortian* (After Allen Forte) and *Shenkarian* (After Alex Schenker) pitch-class notation. This notation is very simple; the tonic for the bass (the note around which the entire progression is based), is notated as $\hat{0}$. Rising semitones will follow notated as

$\hat{1}, \hat{2}, ...$ For example, if the tonic is C, C# is notated as $\hat{1}$ and D is notated as $\hat{2}$. $\hat{t}$ represents the third of a chord. The amount of semitones from the root to the third changes from 3 to 4.

Chords are written as sets in the form of $C = \{A_s, A_a, A_t, A_b\}$. Multiple chords strung together create matrices in the form of

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ A_{31} & A_{32} & \cdots & A_{3m} \\ A_{41} & A_{42} & \cdots & A_{4m} \end{bmatrix}$$

With this, we can begin to establish our rules mathematically.

1. $A_b = \hat{0}$
2. $A_{(n)(m)} - A_{(n)(m-1)} \neq A_{(n_2)(m)} - A_{(n_2)(m-1)} \Leftrightarrow A_{(n)(m)} - A_{(n)(m-1)} = \hat{7}, \hat{12}$
3. $A_s > A_a > A_t > A_b$
4. $\exists A_{root} | \hat{12} \in C$
5. $\exists A_{third} - \hat{t} | 12 \in C$
6. $\#\{C - \hat{7} | \hat{12}\} \leq 1$
7. $A_n - A_{n-1} \leq \hat{12}$

Now that the rules have been defined, we can begin answering the problem at hand. The larger problem can be simplified at first to "How many permutations of a single chord" can we create." If time will allow, I will try to solve the main problem. But for now, we must focus on the more specific problem.

The question of how many ways a single chord can be voiced is extremely simple, although it took me an embarrassingly long time to think of. The bass only has one option. The next voice has 3 options. The last 2 both have two options as the fifth can only have one voice sounding it. Then we must subtract 1 to remove the permutation of all 4 voices playing the root (as the third must also be sounded). That leaves us with the simple result of $2^2 \cdot 3 - 1 = 11$. The third rule is accounted for here without us even trying, as because voices do not need to be within their own range, and voices cannot extend past an octave of one another (Rule 6), all voices have only 1 eligible pitch class. This leads us to the harder problem of analyzing the movement between harmonies.

The first rule to enact is the most important one in voice leading, that is, no parallel Fifths nor Octaves. This rule is written somewhat clunkily as Rule 2.

$$A_{(n)(m)} - A_{(n)(m-1)} \neq A_{(n_2)(m)} - A_{(n_2)(m-1)} \Leftrightarrow A_{(n)(m)} - A_{(n)(m-1)} = \hat{7}, \hat{12}$$

In essence, it states that if two voices are separated by exactly 7 or 12 semitones, they cannot move by the same amount. Thus, the Biad C-G, could not move to biad D-A. We know from the first problem that the probability of having one

$\hat{7}$ is $\frac{1}{2}$. But because there are only 3 options to fit 4 boxes, the probability of having at least one octave is 1. Now we can begin to draw a complex tree of contingencies.

If Fifth we must move 2 notes We give this preference, we have 3 choices for that note, and 2 for the second note.

If Octave we move one note. One note is already bounded to where it is, so we have two options.

We are once again given 11.

Therefore, if the first chord is 11, and there 11 permutations for the following, correcting for over-counting, there may be 11 for the following.

$$11 \cdot 11 \cdot 11 \cdots$$

The solution to amount of permutations of harmonizations to a series of triad chords, as far as I can determine, is simply 11^n, where n represents the number of chords in the sequence