A Tutorial on Catalan Objects

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July 11, 2021

1 Introduction

The Catalan number sequence is ubiquitous, as it is the solution to many combinatorial problems. In fact, the Catalan numbers have 214 different combinatorial interpretations, related to paths on the coordinate plane, strings, graphs, trees, partitions, pattern-avoiding permutations, and more. [Sta]

This paper will focus on 6 of the 214 different interpretations. It will show why these are interpretations of the Catalan numbers, through proofs that either form bijections with other interpretations or leverage the interesting properties of the Catalan numbers themselves. It will give insight as to how and why the remaining 208 interpretations are interpretations of the Catalan numbers.

2 Catalan Numbers and Dyck Paths

Definition 2.1. The Catalan number sequence is a special sequence of numbers, much like the Fibonacci numbers. Given a natural number n, the n^{th} Catalan number is defined as $\frac{1}{n+1}\binom{2n}{n}$, and it is denoted as C_n.

Here are the first few terms of the Catalan number sequence:

$$1, 1, 2, 5, 14, 42, 132, 429, ...$$

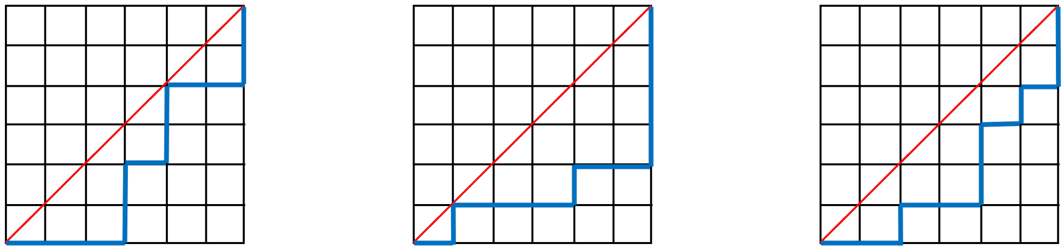
An example of a combinatorial object that is counted by the Catalan numbers is Dyck paths.

Definition 2.2. A **Dyck path** is a path on the 2-dimensional coordinate plane starting at $(0, 0)$ and ending at (n, n) such that:

- the path consists of rightward steps of length 1 and upward steps of length 1 that are parallel to the coordinate axes
- every point on the path is not above the line $y = x$

Note: In this paper, "a Dyck path from $(0, 0)$ to (n, n)" will be used interchangeably with "a Dyck path of length $2n$".

Example 2.2.1. Here are some examples of Dyck paths of length 12 (from $(0, 0)$ to $(6, 6)$):



(a) (b) (c)

Figure 1: Examples of Dyck paths of length 12 ($n = 6$).

The red line is the $y = x$ line, under which the Dyck paths are drawn in blue.

Theorem 2.1. [1] *The total number of Dyck paths from* $(0, 0)$ *and* (n, n) *is equal to* C_n*.*

Here is another combinatorial object counted by the Catalan numbers: Dyck words. We will see in the proof of Theorem 2.2 that they are in fact very similar to Dyck paths, but they are only different in representation:

Definition 2.3. A **Dyck word** of length $2n$ is a string of n open and n closed parentheses such that, at any point in the string, the number of open parentheses are at least as many as the number of closed parentheses before that point.

Example 2.3.1. Here is an example of a Dyck word of length 8:

”(()(()))”

Let's split this string at an arbitrary point into two strings:

”(()” | ”(()))”

Observing the first substring, we see that the number of open parentheses is greater than or equal to the number of closed parentheses. This will be true when we divide the Dyck word at any point.

Theorem 2.2. *The number of Dyck words of length* $2n$ *is equal to* C_n*.*

Proof. Since we know that there are C_n Dyck paths that go from $(0, 0)$ to (n, n), we can show that there is a bijection between these Dyck paths and Dyck words of length $2n$.

The bijection: We start with an empty string “”, and, as we traverse the given Dyck path from $(0, 0)$ to (n, n), we append characters to the string as follows:

- if we take a rightward step, add an open parenthesis “(” to the string
- if we take a upward step, add a closed parenthesis “)” to the string

[1]Refer to Theorem 5.2 of Chapter 1 in [Rub] for the proof

We are guaranteed to get a Dyck word of length $2n$ using this process. This is because of the inherent characteristic of a Dyck path that it does not go over the line $y = x$. As a result, if we were traversing or walking along a Dyck path, the y-coordinate of our location (the number of upward steps we have taken) has to be less than or equal to the x-coordinate of our location (the number of rightward steps we have taken). So, after converting a Dyck path to a string of open and closed parentheses, we see that, at any given point in that string, the number of open parentheses before that point is at least as many closed parentheses before that point. Hence, the string that we obtain is a Dyck word by definition.

Our process for converting Dyck paths to Dyck words is invertible; we can read a given Dyck word from left to right and then build a Dyck path as follows:

- if we encounter a "(", take a rightward step
- if we encounter a ")", take an upward step

To demonstrate how the process of converting Dyck paths to Dyck words works, let's take the Dyck path in Figure 1b and convert that into a Dyck word:

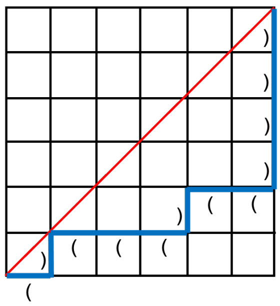


Figure 2: Dyck path labeled with '('s and ')'s.

Each rightward step is labeled with a "(" and each upward step is labeled with a ")". From this path, we get the Dyck word

$$()((()(())))$$

We can also use this string to recover the original Dyck path.

Here, we have shown that we can produce a Dyck word from a Dyck path, and we can produce a Dyck path from a Dyck word. Hence, this process forms a bijection between the Dyck paths and Dyck words.

Therefore, the total number of Dyck words of length $2n$ is equal to C_n. ■

Next, we are going to look at another combinatorial object counted by the Catalan numbers that are related to partitions. But, first, a few definitions are required to describe our Catalan object:

Definition 2.4. An **ordered partition** P of n is a sequence of k positive integers $(p_1, p_2, p_3, ..., p_k)$ such that $p_1 + p_2 + p_3 + ... + p_k = n$ and $1 \leq k \leq n$. Note that the order of the numbers matters.

Definition 2.5. Let a **good pair of partitions** of n be a pair (P, Q) of ordered partitions of n such that they have the same number of elements/addends. Another constraint on this pair is that, for any integer t such that $1 \leq t \leq k$, the sum of the first t elements of P is always greater than or equal to the sum of the first t elements of Q. In other words,

$$\sum_{i=1}^{t} p_i \geq \sum_{i=1}^{t} q_i$$

when $1 \leq t \leq k$.

Now, we claim that the "good pairs of partitions" is counted by the Catalan numbers:

Theorem 2.3. *The total number of "good pairs of partitions" of n is equal to C_n.*

Proof. Just as we have done in the proof of Theorem 2.2, we can construct a bijection between the Dyck paths of length $2n$ and the good pairs of partitions of n.

The bijection: Using the $2n$ numbers (all the p_i's and q_i's) in a given good pair of partitions (P, Q), we can trace out and traverse a Dyck path as follows:

- take p_1 rightward steps
- take q_1 upward steps
- take p_2 rightward steps

  .
  .
  .
- take p_k rightward steps
- take q_k upward steps

This process ensures that we get a Dyck path. First, we know that we get a path that starts at $(0, 0)$ and ends at (n, n) because the total number of rightward steps taken is $\sum_{i=0}^{k} p_i$, which is equal to n, and the total number of upward steps taken is $\sum_{i=0}^{k} q_i$, which is also equal to n. But, we also have to prove that our path never goes above the line $y = x$.

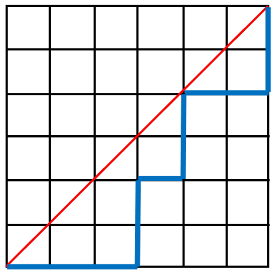
To prove this, let's look at the following constraint on pairs of partitions, given in Definition 2.5:

$$\sum_{i=1}^{t} p_i \geq \sum_{i=1}^{t} q_i, \text{ when } \ 1 \leq t \leq k.$$

Notice that $\sum_{i=1}^{t} p_i$ represents the number of rightward steps we have taken take before an arbitrary point, and that $\sum_{i=1}^{t} q_i$ represents the number of upward steps we have taken before that point. So, this inequality constraint implies that, at any point on the path, the number of rightward steps we have taken is greater than or equal to the number of upward steps we have taken. This means that the path that we trace out does not go over the line $y = x$, and hence it is a Dyck path.

This process is invertible. Given any Dyck path, we can find stretches of rightward steps and stretches of upward steps in the path, and then we can count the length of each stretch. The lengths of the rightward stretches are placed in the sequence P, and the lengths of the upward stretches are placed in the sequence Q. For the same reason that we get a Dyck path from a good pair of partitions, we will get a good pair of partitions (P, Q) from our Dyck path. Let's see how this works using an example.

Let's take the Dyck path in Figure 1a:



To get a good pair of partitions, we can first mark the lengths of the stretches of rightward steps and upward steps:

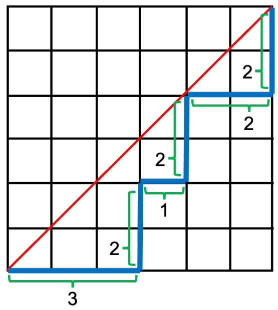


Figure 3: Dyck path with labeled stretches of steps

The lengths of the rightward stretches should be placed in the sequence P, and the lengths of the upward stretches should be placed in the sequence Q:

$$P = (3, 1, 2)$$
$$Q = (2, 2, 2)$$

Hence, we get (P, Q), which is a good pair of partitions. We can verify that they are "good" by testing whether the inequality constraint imposed on good pairs of partitions holds for (P, Q):

$$3 \geq 2$$
$$3 + 1 \geq 2 + 2$$
$$3 + 1 + 2 = 2 + 2 + 2$$

We can also recover the original Dyck path from this "good pair of partitions".

Here, we have shown that we can produce a Dyck path from a good pair of partitions, and we can produce a good pair of partitions from a Dyck path. Hence, we found a bijection between the Dyck paths and the good pairs of partitions.

Therefore, the total number of good pairs of partitions of n is equal to C_n. ■

Now that we have visited some combinatorial objects that are counted by the Catalan numbers, now is an appropriate time to define what **Catalan objects** are:

Definition 2.6. A combinatorial object is a **Catalan object** if and only if, for all n, there are exactly C_n instances of such a combinatorial object with n elements.

Example 2.6.1. The set of Dyck paths is a **Catalan object** because, for all n, there are C_n Dyck paths with n rightward steps and n upward steps.

3 Catalan Objects and a Neat Recurrence

Theorem 3.1. [2] *The Catalan numbers follow this recurrence relationship:*

$$\begin{aligned} C_n &= \sum_{k=0}^{n-1} C_k C_{n-k-1} \\ &= C_0 C_{n-1} + C_1 C_{n-2} + ... + C_{n-2} C_1 + C_{n-1} C_0 \end{aligned}$$

This recurrence relationship has a very interesting consequence. It means that we can construct a Catalan object of size n by concatenating two Catalan objects with sizes that add up to $n-1$.

To explain further, the LHS of Theorem 3.1 counts the number of Catalan objects of size n. The sum on the RHS counts the total number of ways to concatenate:

- a Catalan object of size 0 with a Catalan object of size $n-1 \rightarrow C_0 C_{n-1}$
- a Catalan object of size 1 with a Catalan object of size $n-2 \rightarrow C_1 C_{n-2}$

  .
  .
  .
- a Catalan object of size $n-2$ with a Catalan object of size $1 \rightarrow C_{n-2} C_1$
- a Catalan object of size $n-1$ with a Catalan object of size $0 \rightarrow C_{n-1} C_0$

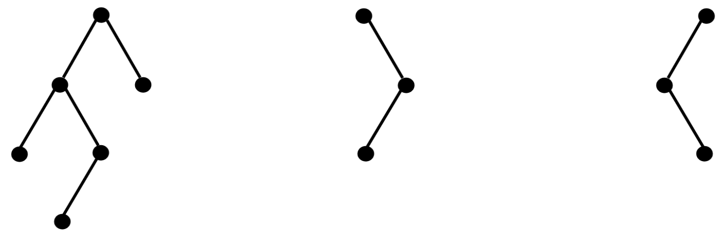
We will see how this principle works in Theorem 3.2 using another Catalan object: binary trees.

Definition 3.1. An **unlabeled rooted binary tree** with n vertices is an undirected graph of n vertices which satisfies the following requirements:

- it is acyclical
- it has one defined root vertex
- every vertex has either:
  - no children
  - a left child
  - a right child
  - a left child and a right child
- none of its vertices are labeled with numbers

[2]Refer to Theorem 3.1 of Chapter 8 in [Rub] for the proof

Example 3.1.1. Here are some examples of unlabeled rooted binary trees:



(a) (b) (c)

Figure 4: Examples of distinct unlabeled rooted binary trees.

Note that the kind of children that each vertex has (having either no children, a left child, a right child, or left & right children) matters.

Theorem 3.2. *The total number of unlabeled rooted binary trees with n vertices is C_n.*

Proof (see [Har19]). Let B_n be the number of unlabeled rooted binary trees with n vertices. To show that $B_n = C_n$ for all $n \geq 0$, we can use strong induction. This entails proving:

1. **the base cases**: $B_0 = C_0$ and $B_1 = C_1$.
2. **the induction step**: if $B_k = C_k$ for all values of $k < n$, then $B_n = C_n$.

Base case: We know that the base cases are true. The number of unlabeled rooted binary trees with 0 vertices is 1, which is equal to C_0. Also, the number of unlabeled rooted binary trees with 1 vertex is 1, which is equal to C_1.

Induction step: To prove that the induction step is true, we will show that the number of unlabeled rooted binary trees follows the same recurrence relationship as the Catalan numbers.

Given the numbers n and k, let's say that we have a rooted binary tree T_k with k vertices and a rooted binary tree T_{n-k-1} with $n-k-1$ vertices. We can "concatenate" these trees together to form a tree with n vertices as follows: let us draw another vertex and designate that to be the root of our new n-vertex tree. Then, we can extend two edges from that vertex so that they connect to the root vertices of T_k and T_{n-k-1}.

Let's demonstrate this process when $n = 10$. We can take two separate binary trees with $k = 6$ vertices and $n - k - 1 = 3$ vertices

(a) A tree with 6 vertices. (b) A tree with 3 vertices.

Figure 5

and we can combine them to form a tree with 10 vertices as shown below:

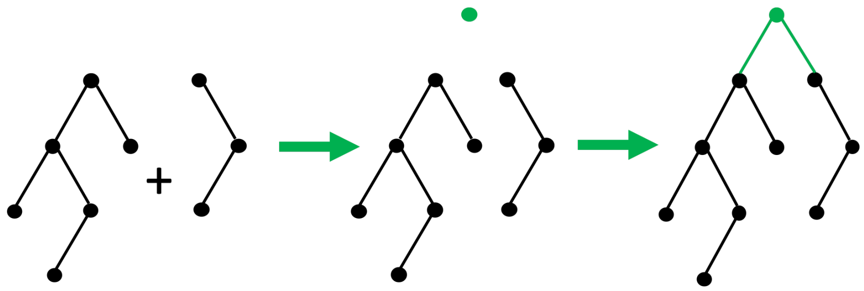


Figure 6: Constructing a tree with 10 vertices using trees with 6 and 3 vertices.

In general, the process gives us unique rooted binary trees. This is because, if we modify our two sub-trees, our process will create a different tree.

So, we can generate all distinct binary trees with n vertices by combining all trees with k vertices with all trees with $n-k-1$ vertices, and then repeating this process for all values of k. For a given k, we know that the number of trees with k vertices is B_k and the number of trees with $n-k-1$ vertices is B_{n-k-1}, so the total number of ways we can "concatenate" these trees together is $B_k \cdot B_{n-k-1}$. If we sum this quantity for all values of k ($0 \leq k \leq n-1$), we get the following recurrence relationship:

$$B_n = \sum_{k=0}^{n-1} B_k B_{n-k-1} \tag{1}$$

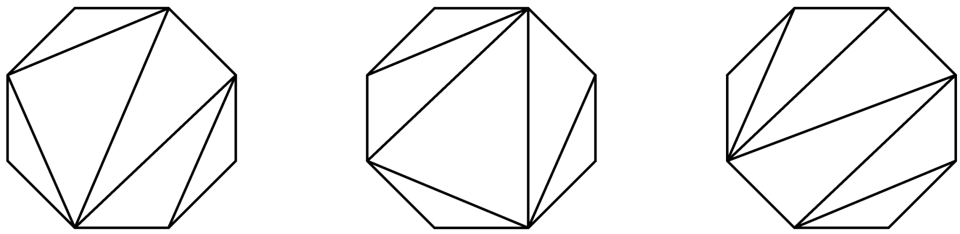
This is the exact recurrence relationship as the Catalan numbers! So, if we assume that $B_k = C_k$ for all $k < n$, we know that B_n will be equal to C_n. Thus, we have proven the induction step.

Since both the base cases and the induction step are true, it is true that $B_n = C_n$ for all $n \geq 0$. ■

We can use the same technique to show that our next combinatorial object is a Catalan object:

Definition 3.2. Given a regular polygon with n sides, a **triangulation** of this polygon is a division of this polygon into $n-2$ triangles whose vertices are the vertices of the polygon. These triangles must cover the whole polygon, and they must not overlap each other. Note that the orientation of the triangulation matters.

Example 3.2.1. Here are a few examples of triangulations of octagons:



(a) (b) (c)

Figure 7: Examples of triangulations of octagons ($n = 8$).

Theorem 3.3. *The total number of triangulations of an $(n+2)$-sided regular polygon is equal to C_n.*

Proof. Let T_n be the number of triangulations of an n-gon. We can show that $T_{n+2} = C_n$ using strong induction, which involves proving the base cases and proving the induction step.

Base case: $T_2 = C_0$ and $T_3 = C_1$
We know that this is true since the number of triangulations of a 2-sided polygon (a line segment) is 1, and the number of triangulations of an equilateral triangle is 1.

Induction step: If $T_{k+2} = C_k$ for all $k < n$, then $T_{n+2} = C_n$
Let there be an $(n+2)$-gon, and let's label all of its $n+2$ vertices counterclockwise using the integers from 1 to $n+2$. Then, let's take any edge of the $(n+2)$-gon, for example, the edge that connects vertex 1 and vertex $n+2$.

An observation is that, in any triangulation of the $(n+2)$-gon, that edge must be part of a single triangle in the triangulation. The proof of this statement is twofold: we need to show that that edge belongs to least one triangle *and* at most one triangle. At least one triangle has to have that edge as a side because, otherwise, the triangulation would not be complete. And, at most one triangle has to have that edge as a side because, if the triangulation had two different triangles that share that edge, those two triangles would overlap. Hence, the edge that connects vertex 1 and $n+2$ must be part of one and only one triangle.

Since we have shown that there is only one triangle that contains vertices 1 and $n+2$, it remains to determine the 3^{rd} vertex of the triangle so that we can fully construct that triangle and eventually build a triangulation. That 3^{rd} vertex could be any of the remaining n vertices: vertices 2, 3, ..., and $n+1$. For example, if our $(n+2)$-gon were a pentagon, we can list out the possible ways to form this triangle, as shown below:

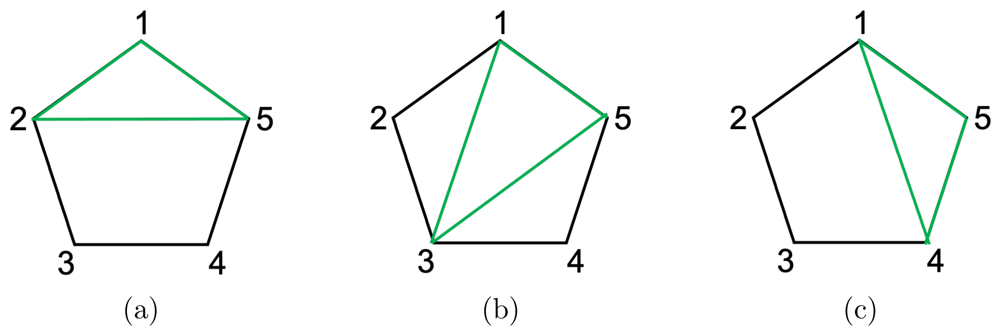


Figure 8: Examples of triangles that contain the edge 1–5 in a pentagon.

Let's say that the 3^{rd} vertex of this triangle is vertex k. The interesting thing about our new triangle is that it divides the $(n+2)$-gon into two new convex polygons: one polygon with vertices 1, 2, ..., $k-1$, and k, and another polygon with vertices k, $k+1$, ..., $n+1$, and $n+2$. For example, if we were using the setup in Figure 8b, when $n+2 = 5$ and $k = 3$, here's how the $(n+2)$-gon would be divided by the triangle into two different polygons:

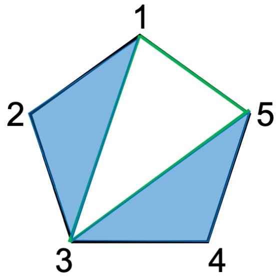


Figure 9: Shaded areas indicate the interiors of the smaller polygons.

Let's also look at the setup in Figure 8a, when $n+2=5$ and $k=2$. Though it may look like the triangle does not split the $(n+2)$-gon into two separate polygons, the triangle does actually split it into two polygons: one with vertices 1 and 2 (a line segment), and one with vertices 2, 3, 4, 5 (a quadrilateral), as shown below:

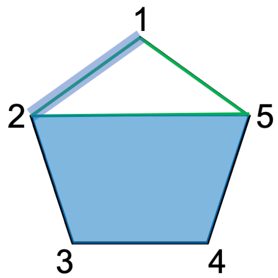


Figure 10: Shaded areas indicate the interiors of the smaller polygons.

So, in general, in order to triangulate the whole $(n+2)$-gon, we must individually triangulate these two smaller convex polygons. We cannot construct triangles that run between these two polygons since they would overlap with the triangle that we have just constructed (the green triangle in Figure 9).

Therefore, if we are given an $(n+2)$-gon and a value for k, we can calculate how many triangulations exists with those constraints as follows. The first smaller polygon has k vertices, so the number of ways to triangulate that polygon is T_k. The second smaller polygon has $n-k+3$ vertices, so the number of ways to triangulate that polygon is T_{n-k+3}. Hence, for a given k, the number of triangulations of the $(n+2)$-gon is $T_k \cdot T_{n-k+3}$. Summing this over all values of k ($2 \leq k \leq n+1$), we get that the total number of triangulations of the $(n+2)$-gon is

$$T_{n+2} = \sum_{k=2}^{n+1} T_k T_{n-k+3} \tag{2}$$

This is very close to the Catalan number recurrence, but it still needs a few more changes. We should replace k with $k+2$, and, as a result, we should modify the bounds of the summation from $(2, n+1)$ to $(0, n-1)$ to preserve the value of the summation. After these changes, we get the recurrence relationship

$$T_{n+2} = \sum_{k=0}^{n-1} T_{k+2} T_{n-k+1}$$

$$\implies T_{n+2} = \sum_{k=0}^{n-1} T_{k+2} T_{(n-k-1)+2} \tag{3}$$

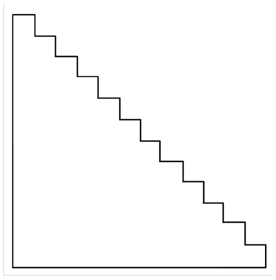
This sequence follows the same recurrence as the Catalan numbers. So, if we assume that $T_{k+2} = C_k$ for all $k < n$ (this implies that $T_{(n-k-1)+2} = C_{n-k-1}$), then we see that $T_{n+2} = C_n$. Thus, we have proven the induction step.

The base cases are true, and the induction step is true. Therefore, for all n, $T_{n+2} = C_n$. ■

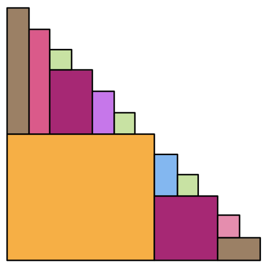
Let's visit one more of the many interesting Catalan objects:

Definition 3.3. Let there be a Ferrer's board with n columns of heights $(n, n-1, n-2, ..., 2, 1)$ (this will be referred to as a triangular Ferrer's board of size n). A **staircase tiling** of size n is a tiling of this board with n rectangles whose sides are parallel to the sides of the Ferrer's board.

Example 3.3.1. Here is a triangular Ferrer's board of size $n = 12$ and an example of a staircase tiling of that board:



(a) Empty triangular Ferrer's board of size 12.



(b) A staircase tiling of size 12 (taken from [For+13])

Figure 11

Theorem 3.4. *The total number of staircase tilings of size n is C_n.*

This theorem will be left as an exercise for the reader, and here is a guiding question that will help you prove it: how can you form new staircase tilings of size n by "concatenating" two smaller staircase tilings whose sizes add up to $n - 1$? Specific to Figure 11b, do you see two smaller staircase tilings whose sizes add up to 11?

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