Division, Combinatorically

Eugenie Cha

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1 Introduction

In this paper, we will prove divisibility by two and three without using the Axiom of Choice. We will do so by showing that for sets A and B, if there is a one-to-one correspondence between $\{0,1\} \times A$ and $\{0,1\} \times B$ or $\{0,1,2\} \times A$ and $\{0,1,2\} \times B$, then there is a one-to-one correspondence between A and B [CD94]. We will use the notation $\asymp$ to denote that there is a one-to-one correspondence.

If we allow ourselves to use the Axiom of Choice, it simplifies the problem significantly. However, we will refrain from using it in this paper as it doesn't produce a well-defined correspondence.

The Axiom of Choice: For a collection of non-empty sets, C, it is always possible to select an element. In other words, we can define a *choice function* $f(x)$ such that $f(S)$ returns an element of S. [Axi]

When we apply the Axiom of Choice to our problem, we select an element from A and B and define a choice function between the two. We then derive our correspondence from our choice function.

For example, let's define $A = \{1,2,3\}$ and $B = \{2,3,4\}$.

If we choose a set C within A and D within B, we can define a function. For example, $C = \{1,2\}$ and $D = \{2,4\}$. We can define a choice function such that: $f(1) = 2$ and $f(2) = 4$, where the inputs are elements of C and the outputs are elements of D.

The problem with using the Axiom of Choice is that it doesn't provide us with a unique function. In other words, there isn't a specific rule that we can apply in order to form a bijection between the two sets. Therefore, providing us with a function that isn't well defined. Thus, we will prove combinatorial divisibility by two and three without using the Axiom of Choice.

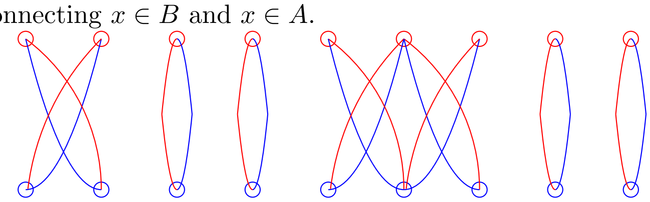
2 Background

2.1 The Cantor-Schröder-Bernstein Theorem

The Cantor-Schröder-Bernstein Theorem (abbreviated to CSB) uses the notation $\preceq$, which means an injection into. For example, $A \preceq B$ means there's an injection from A to B. CSB also uses the $\asymp$ notation we discussed before.

Cantor-Schröder-Berstein Theorem: If $A \preceq B$ and $B \preceq A$, then $A \asymp B$.

Proof: We know $f : A \to B$ and $g : B \to A$. We can visualize the injections by representing A as a set of blue dots with blue arcs connecting $x \in A$ and $x \in B$ and the opposite goes for B: B is a collection of red dots with red arcs connecting $x \in B$ and $x \in A$.



As shown in the graph, there are different patterns: a finite cycle and a doubly-infinite path. The cycles are the most straightforward. They start on one vertex and cycle to a different-colored one on the opposing side. Doubly-infinite paths always start on either A or B and correspond to a point on the opposing side. Thus, all of the vertices are paired up with a different colored one, which means there is a one-to-one correspondence between A and B. ■

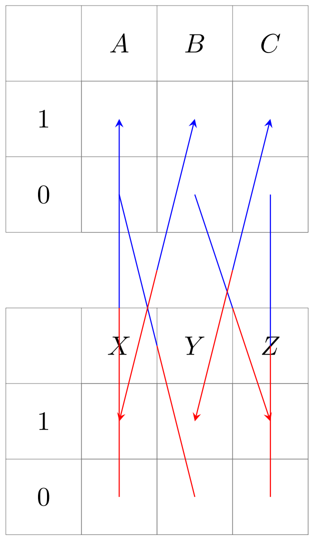
2.2 Swallowing

If $A+B \asymp A$, then A swallows B or $A \gg B$. We could also say $A+B \preceq A$ since by CSB, $A+B \asymp A$ if $A+B \preceq A$ and $A \preceq A+B$. The Hilbert Hotel model (also known as the Roach Hotel model) is what's commonly used to visualize swallowing. Every element in A represents a hotel room, and every element in B represents a new guest coming to the hotel. In order for us to accommodate all the guests, we tell every element in room $f(x)$ to move to room $f(f(x))$, every element in room $f(f(x))$ to move to room $f(f(f(x)))$, and so on so forth. By doing so, we will create rooms for the elements of B to check into without having to kick out elements of A.

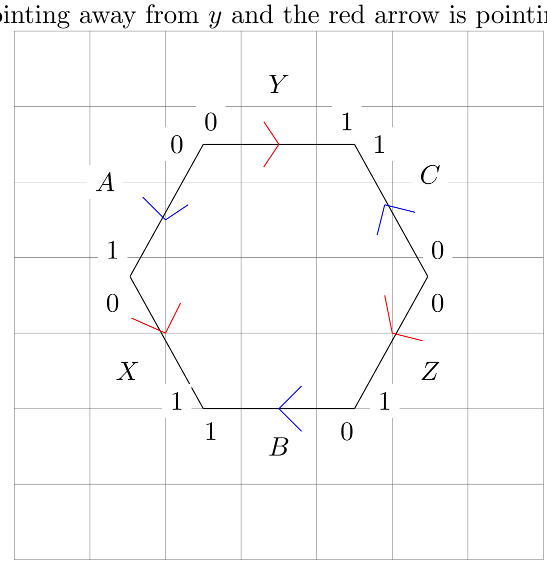
3 Combinatorial Division

3.1 Division by Two

In order to show divisibility by two, we are given the bijection $f : \{0, 1\} \times A \to \{0, 1\} \times B$ and want to show that $g : A \to B$ is a bijective function. Let's say that set A is a collection of blue arrows and set B is a collection of red arrows. Each arrow has a head and a tail. Let's assign the tail of each arrow as a 0 and the head of each arrow as a 1. Each head must correspond with a head of a different color and the same goes for tails. For example, let's say $A = \{a, b, c\}$ and $B = \{x, y, z\}$. The following diagram is a random pair up of $\{0, 1\} \times A$ and $\{0, 1\} \times B$:



We can take this arrangement of arrows and redraw them as a necklace of arrows. For example, $(0, a)$ matches up with $(0, y)$, which means the tail of the blue arrow matches up with the tail of the red arrow. Thus, the blue arrow is pointing away from y and the red arrow is pointing away from a:

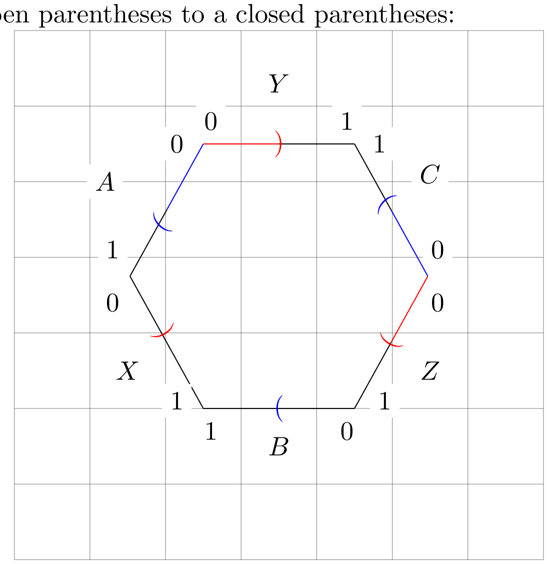


Notice that there is an equal number of red and blue arrows. This means we can try to pair one blue arrow with one red arrow, which is exactly what we want in order to show that there is a bijection between A and B. In order to pair the arrows up, let's imagine that the arrows are pointing in the direction of their pair. For example, the blue arrow for c is pointing to y's red arrow. Thus, the two pair up. We can follow a similar pattern for the other arrows. However, notice that both x and z are pointing to b, but b is only pointing to one of them, so z is left without a pair. The same goes for a and x. Therefore,

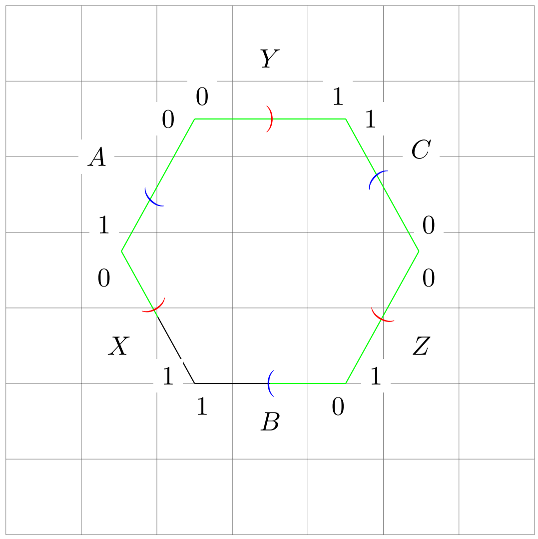
a and z are left without pairs.

One might intuitively think: can't we just pair up a and z since they're both unpaired and have opposite colors? The reason why we can't do this is because we'd be using the Axiom of Choice. By pairing up the unmatched elements, we'd be creating a one-to-one correspondence for that specific case. However, as mentioned in the introduction, this doesn't produce a well-defined correspondence as it changes whenever the orientation of the arrows change.

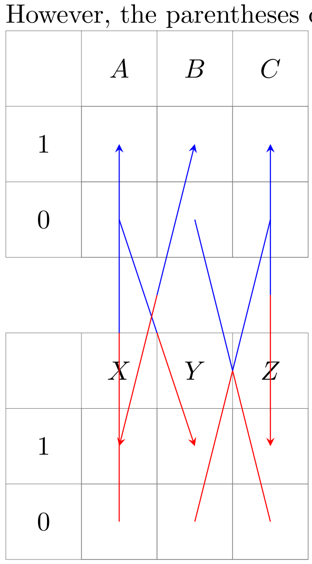
Instead of pairing the unmatched arrows up, we can change the arrows to parentheses to create a diagram that allows every element to have a pair. We're able to change this notation because, unlike pairing the unmatched pairs up, we're creating a well-defined correspondence. By changing to parentheses, we're merely defining a different rule than before. For this case, we can match up every open parentheses to a closed parentheses:



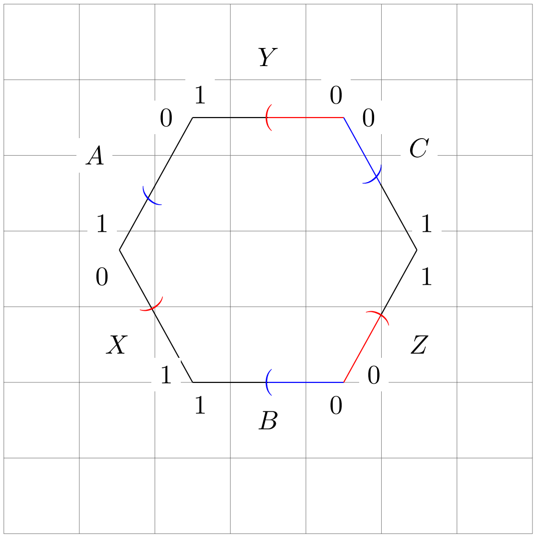
As for x and b, they're matching up as well since they're a pair of open and closed parentheses, except the space between them is larger. The green line in the diagram shows their path:



Thus, we have the correspondence: $g(a) = y$, $g(b) = x$, and $g(c) = z$.
However, the parentheses don't always match up. Here's a correspondence:

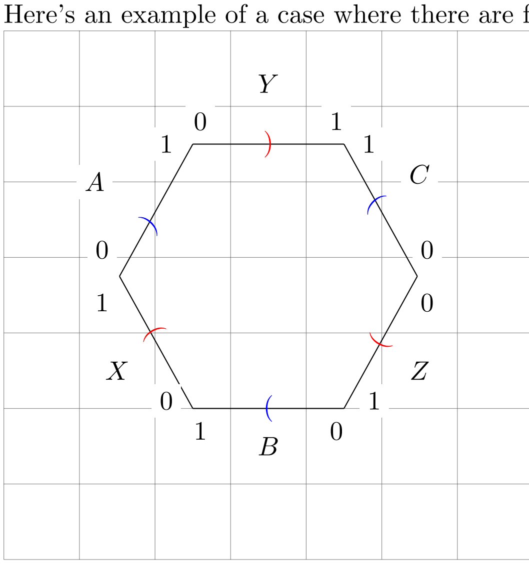


And, the graph:



In this case, we can go back to our original idea of matching the parentheses based off of the direction they're pointing in. Both a and x are pointing in the same direction, counter-clockwise. Since every unmatched pair points in the same direction and the parentheses are in a loop, we can pair them up. We pair every blue parenthesis with a red one that comes before it in terms of the direction the arrows are pointing. It also works to start with a red parenthesis and keep going until we meet a blue one. In this example, we start at a. We go counter-clockwise until we meet an unmatched red parenthesis. In this case, we meet x. Thus, we match them up.

Here's an example of a case where there are four unmatched parentheses:



Since c and z are a set of open and closed parentheses, they pair up. As for the unpaired parentheses, we can apply the same logic as we did in the previous example. We can start at any of the unmatched parentheses. Let's start at a. When we go counter-clockwise, we meet y. Thus, a and y are a pair. Next, we start with b. We go counter-clockwise and meet x. Thus, x and b are a pair:

Therefore, we have our correspondence: $g(c) = z$, $g(a) = y$, and $g(b) = x$

Now, we have a general format for these types of problems. We start off by matching the open and closed parentheses. If there are any unmatched parentheses left over, we start at an unmatched parenthesis and go the direction the arrows are pointing until we meet an unmatched parenthesis with the opposite color. Then, we match the two together.

But, what if we had infinitely many elements in both sets instead of just 3? We can just apply the same rules that we applied for the set of size 3 example. This works because an infinitely long necklace allows for pairs of open and closed parentheses and also parentheses that can direct the remaining parentheses to pair up with one below or above it. Therefore, we can divide by two.

3.2 Division by Three

Now, we want to prove that if there is a bijection $\{0, 1, 2\} \times A \to \{0, 1, 2\} \times B$, then we can produce a bijection $f : A \to B$. In order to pictorially visualize this, let's say that $\{0, 1, 2\} \times A$ and $\{0, 1, 2\} \times B$ are $30° - 60° - 90°$ triangles with 0 as the vertex of $30°$, 1 as the vertex of $60°$, and 2 as the vertex of $90°$. The bijection that we're given can be represented by attaching the verteces of these different colored triangles in different ways. However, the problem we run into is: what if there are leftover triangles of one color? In order to solve this problem, let's take a look at Tarski's Lemma:

Tarski's Lemma: If $A \succeq B$ and $3 \times B \succeq 3 \times A$, then $A \asymp B$.

To prove Tarski's Lemma, consider the following lemmas:

Lemma 1 If $A \ll B$ and $B \ll C$, then $A + B \ll C$.

Proof: We can prove this using Hilbert's Hotel model – introduced in swallowing. $A \ll B$ represents that we can fit a set of guests A. Similarly, we can fit a set of guests B. Combining them together, we can first fit a set of guests A then a set of guests B, which means we can fit both sets. Therefore, $A + B \ll C$.

■

Lemma 2: If $A \ll B_1 + B_2 + B_3$, then we can express A as $A = A_1 + A_2 + A_3$ such that $A_1 \ll B_1$, $A_2 \ll B_2$, and $A_3 \ll B_3$.

Proof: Let's say the B's represent different buildings. The expression $A \ll B_1 + B_2 + B_3$ means that we can accommodate A guests using our three buildings. We claim that we can split A into three groups such that each group can fit into a different building. For each guest that we plan to accommodate, we can assign them into one of the three buildings. For a set of guests who are designated to go to building one, we'll move every guest already at the hotel to their respective $f(x)$ room. We can repeat the same process for guests designated for buildings two and three.

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Lemma 3: If $A \ll 3 \times B$, then $A \ll B$.

Proof: Let's say $A = A_1 + A_2 + A_3$. Similarly, $3 \times B = B + B + B$:

$$A \ll 3 \times B$$

$$A_1 + A_2 + A_3 \ll B + B + B$$

By lemma 2, we can say:

$$A_1 \ll B$$

$$A_2 \ll B$$

$$A_3 \ll B$$

By lemma 1, we can say:

$$A_1 + A_2 \ll B$$

$$A_1 + A_2 + A_3 \ll B$$

$$A \ll B$$

■

We can now use all 3 lemmas in order to prove Tarski's lemma: If $A \succeq B$ and $3 \times B \succeq 3 \times A$, then $A \asymp B$.

Proof: Since $A \succeq B$, we can say $A \asymp B + C$, where C is some set. If we want to prove that $A \asymp B$, it's the same as trying to show that $C \ll B$.

$$3 \times B \succeq 3 \times A$$

$$3 \times B \succeq 3 \times (B + C)$$

$$3 \times B \succeq 3 \times B + 3 \times C$$

$$3 \times B \gg 3 \times C$$

$$3 \times B \gg C$$

$$B \gg C$$

Thus, $A \asymp B$.

■

Now, we can apply Tarski's lemma. The reason why we proved Tarski's Lemma is because we need to consider the case where there are leftover blue or red triangles. If the blue and red triangles connect vertice to vertice perfectly (without any triangles leftover), then we don't have to worry. However, if there's an excess of blue or red triangles, then we run into a problem. Another way to say that we have excess blue and red triangles is $3\times A \gg 3\times B$ (or $3\times A \ll 3\times B$. And by Tarski's lemma, we can say that there exists a bijection $A \asymp B$, which is exactly what we're looking for. Therefore, we can divide by three.

References

[Axi] A home page for the axiom of choice. https://math.vanderbilt.edu/schectex/ccc/choice.html. Accessed: 2021-06-20.

[CD94] John Conway and Peter Doyle. Division by three, 1994.