Hyperplane Arrangements

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1 Abstract

In this paper, we will explore hyperplane arrangements and their relation to combinatorics. Some surprising relationships will appear. We will discuss posets, Characteristic polynomials, Whitney's Theorem, and Zaslavsky's Theorem, among others. As a motivating question, before proceeding, consider how many spaces n lines separates 2 dimensional space into.

2 Introduction to Hyperplane Arrangements

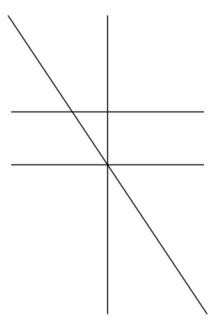
A hyperplane is the generalization of a 2 dimensional plane to higher dimensions, or said more formally

Definition 1 *(hyperplane) For a field K, an $n-1$ dimensional affine* ***hyperplane*** *of K^n is the affine subspace $\{v \in K^n : a \cdot v = b\}$. Or alternatively, a hyperplane in K^n is the set of points in K^n satisfying a linear equation $a_1x_1 + \cdots + a_nx_n = b$, for some $a_1, \cdots, a_n, b \in \mathbb{R}$*

We can use hyperplanes to build hyperplane arrangements, defined as

Definition 2 *(hyperplane arrangement) A* ***Hyperplane Arrangement****, called A, is the union of a finite set of hyperplanes. r(A) is a function for how many regions A divides space into.*

As an example of r(A), let A be the image below of a hyperplane arrangement in $\mathbb{R}^2$, then r(A) is 9.



3 Intersection Posets

Before we can define an intersection poset, we must first define the poset.

Definition 3 *(Poset) A* ***poset*** *is a set along with a binary operation (called $\leq$) where some of the elements in the poset are comparable.*

We can relate posets to hyperplane arrangements by the intersection poset.

Definition 4 *(Intersection Poset) The* ***intersection poset*** *of an arrangement A, denoted L(A), as the set of all non-empty intersections of sets of hyperplanes ordered by reverse inclusion, meaning that for 2 intersections I_1 and I_2, $I_1 \leq I_2$ iff $I_2 \subset I_2$*

There are several poset functions that are used on hyperplanes, such as the Mobius function

Definition 5 *(Mobius Function) Let P be a finite poset. Define a function $\mu = \mu_P : Int(P) \to Z$, called the* ***Mobius function*** *of P, by: $\mu(x, x) = 1$, for all $x \in P$ $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$, for all $x < y$ in P*

4 Characteristic Polynomials

A characteristic polynomial is a way of representing a hyperplane arrangement in polynomial form, like a generating function.

Definition 6 *(Characteristic Polynomial) The* ***characteristic polynomial*** *$\chi_A(t)$ of the arrangement A is defined by*

$$\chi_A(t) = \sum_{x \in L(A)} \mu(x) t^{dim(x)}$$

There are several interesting properties of the characteristic polynomial. Let us look at 2 recurrences related to it.

Theorem 1 *(Hyperplane Recurrence)*
Let (A, A' , A") be a triple of real arrangements with distinguished hyperplane H'. Then $r(A') = r(A') + r(A'')$

As a side note, if $rank(A) = rank(A')$, then also $rank(A) = 1 + rank(A'')$. Let us give a proof of this recurrence. Note that $r(A)$ equals $r(A')$ plus the number of regions of A' cut into two regions by H'. Let R' be such a region of A' . Then $R' \cap H' \in R(A'')$. Conversely, if $R'' \in R(A'')$ then points near R'' on either side of H0 belong to the same region $R' \in R(A')$, since any $H \in R(A')$ separating them would intersect R'' . Thus R' is cut in two by H'. We have established a bijection between regions of A' cut into two by H' and regions of A'' , establishing the first recurrence.

Now that we have established that recurrence, we come to perhaps the more famous recurrence

Theorem 2 *(Deletion Restriction) Let (A, A', A'') be a triple of real arrangements. Then $\chi_A(x) = \chi_{A'}(x) - \chi_{A''}(x)$.*

5 Special Arrangements

Some Arrangements of hyperplanes are notable, and we will discuss a few special arrangements in this section. The first special arrangement is the braid arrangement.

Definition 7 *(Braid Arrangement) the **braid arrangement** is defined as $B_n = \{x_i - x_j = 0 | i \neq j\}$*

B_n has $\binom{n}{2}$ hyperplanes. To count the number of regions when K = R, note that specifying which side of the hyperplane $x_i - x_j = 0$ a point $(a)1, \cdots, a_n)$ lies on is equivalent to specifying whether $a_i < a_j$ or $a_i > a_j$. This means that the number of regions is the number of ways that we can specify whether $a_i < aj$ or $a_i > a_j$ for $1 \leq i < j \leq n$

The other famous special arrangements are all extensions of the braid arrangement in some way. For example, in the Shi arrangement, $x_i - x_j = 0, 1$ and in the Catalan arrangement, $x_i - x_j = -1, 0, 1$.

6 Zaslavsky's Theorem

Zaslavsky's Theorem is one of the major theorems in the number of regions in Hyperplane Geometry.

Definition 8 *(regions) For an arrangement A in R^n , define the number of **regions**, denoted r(A), to be the number of connected components of $R^n - \cup_{H \in A} H$. Similarly, define b(A) as the number of **relatively bounded regions** of of A.*

These two definitions are used in Zaslvasky's Theorem

Theorem 3 *(Zaslvasky's Theorem)*

$$r(A) = (-1)^n \chi_A(-1)$$

and

$$b(A) = (-1)^{rank(A)} \chi_A(1)$$

A proof of Zaslavsky's Theorem: this holds for $A = \emptyset$, since $r(\emptyset) = 1$ and $\chi\emptyset(t) = tn$. Both $r(A)$ and $(-1)^n \chi_A(-1)$ satisfy the same recurrence, so the proof follows. Now consider the equation. Again it holds for $A = \emptyset$ since $b(\emptyset) = 1$. (Recall that b(A) is the number of relatively bounded regions. When $A = \emptyset$, the entire ambient space R n is relatively bounded.) Now$\chi_A(1) = \chi_{A'}(1) - \chi_{A''}(1)$. Let $d(A) = (-1)^{rank(A)} \chi_A(1)$. If $rank(A) = rank(A') = rank(A'') + 1$, then $d(A) = d(A') + d(A'')$. If $rank(A) = rank(A') + 1$ then $b(A) = 0$ and $L(A') \cong L(A'')$. Thus we have $d(A) = 0$. Hence in all cases $b(A)$ and $d(A)$ satisfy the same recurrence, so $b(A) = d(A)$.

6.1 Whitney's Theorem

Another good way that characteristic polynomials come into use with hyperplanes is with Whitney's Theorem. It gives an alternate expression of how to write a characteristic polynomial.

Theorem 4 *(Whitney's Theorem) Let A be an arrangement in an n-dimensional vector space. Then*

$$\chi_A(t) = \sum_{B \subset A, B central} (-1)^{\#B} t^{n-rank(B)}$$

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In Whitney's Theorem, the sum is taken over all sets of hyperplanes B of A with a nonempty intersection. Now that we have shown Whitney's Theorem, we will write a proof:

The proof of this theorem requires another theorem, the Crosscut Theorem,

Theorem 5 *(the Cross-Cut Theorem) Let L be a finite lattice. Let X be a subset of L such that such that if $y \in L, y \neq \hat{0}$, then some $x \in X$satisfies $x \leq y$. Let N_k be the number of k-element subsets of X with join $\hat{1}$. Then $\mu_L(\hat{0}, \hat{1}) = N_0 - N_1 + N_2 - \cdots$*

Now that we have the Crosscut Theorem, we can use that to prove Whitney's Theorem.

Let $z \in L(A)$. Let $\Lambda_z = x \in L(A) : x \leq z$, the principal order ideal generated by z. Recall the definition $A_z = \{H \in A : H \leq z(i.e., z \subseteq H)\}$. By the Crosscut Theorem , we have $\mu(z) = \sum_k (-1)^k N_k(z)$, where $N_k(z)$ is the number of k-subsets of A_z with join z. In other words, $\mu(z) = \sum_{B \subseteq Azz = TH \in BH} (-1)^{\#B}$. Note that $z = \cap_{H \in B} H$ implies that $rank(B) = n - dim(z)$. Now multiply both sides by $t^{dim(z)}$ and sum over z to obtain the equation

7 Conclusion

Hyperplane arrangements are a fascinating area of study. They appear to be about geometry, but also involve the tools of combinatorics. There are many subjects in hyperplane arrangements which I did not cover in this paper, such as Shi and Catalan arrangments.

References

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