CATALAN NUMBERS

ASHWIN RAJAN

Abstract. Catalan numbers are some of the most ubiquitous numbers, appearing in many ostensibly unrelated combinatorial problems. In this expository paper, we will discuss the Catalan numbers from both an algebraic and a combinatorial perspective. We will begin by introducing their explicit formula and recurrences. Then, we will move into a discussion of some of the counting problems Catalan numbers solve, either forming bijections between different problems or showing that the problem satisfies the Catalan numbers' recurrence relation and initial conditions. Finally, we will discuss the generating function and growth rate of Catalan numbers.

1. Algebraic Introduction

Definition 1.1 (Catalan numbers). *Catalan numbers* are numbers of the form $C_n = \frac{1}{n+1}\binom{2n}{n}$ for nonnegative n.

We can immediately note that the binomial coefficient in the explicit formula indicates that the Catalan Numbers will appear in combinatorial problems. Although we can glean this important information from the explicit formula, the explicit formula is not very useful for much more than calculating Catalan numbers for large n. In fact, more useful for our discussion of the Catalan Numbers are its Recurrence Relations.

Theorem 1.2 (First Recurrence Relation). *Given the initial condition $C_0 = 1$,*

$$\sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

Many combinatorial proofs for the Recurrence Relation exist, with one prominent one using Dyck paths. However, we can also prove this recurrence relation purely algebraically using the explicit formula for the Catalan numbers and a helper function $a(n, j)$.

Proof of Theorem 1.2. We define

$$a(n, j) := \frac{2j-n}{2n(n+1)}\binom{2j}{j}\binom{2n-2j}{n-j}.$$

We claim that $a(n, i+1) - a(n, i) = C_i C_{n-i-1}$. Before we evaluate the left hand side, we can expand the right hand side. By doing so, we know into which form we want to mold the left hand side. We have

$$C_i C_{n-i-1} = \frac{1}{i+1}\binom{2i}{i}\frac{1}{n-i}\binom{2n-2i-2}{n-i-1}.$$

Now, we can move into simplifying our left hand side, keeping in mind that we seek to manipulate it to be in form of the expression above.

Date: July 2021.

Using our formula for $a(n,j)$, we get that the left hand side is equivalent to

$$a(n,i+1)-a(n,i)=\frac{2i+2-n}{2n(n+1)}\binom{2i+2}{i+1}\binom{2n-2i-2}{n-i-1}-\frac{2i-n}{2n(n+1)}\binom{2i}{i}\binom{2n-2i}{n-i}.$$

Factoring out common factors and simplifying, we get

$$=\frac{1}{2n(n+1)}\binom{2n-2i}{n-i}\binom{2n-2i-2}{n-i-1}\left(\frac{4i+2}{i+1}(2i+2-n)-\frac{2n-2i-2}{n-i}(2i-n)\right).$$

We notice that the denominators $i+1$ and $n-i$ are also on the right hand side of the equation we are trying to prove, so we factor them out, getting

$$=\frac{\binom{2n-2i}{n-i}\binom{2n-2i-2}{n-i-1}}{2n(n+1)(i+1)(n-i)}\left((2i+2)(2i+2-n)(n-i)-(4n-4i-2)(2i-n)(i+1)\right).$$

Simplifying the expression yields

$$=\frac{\binom{2n-2i}{n-i}\binom{2n-2i-2}{n-i-1}}{2n(n+1)(i+1)(n-i)}\left(2n(n+1)\right)=\frac{\binom{2n-2i}{n-i}\binom{2n-2i-2}{n-i-1}}{(i+1)(n-i)},$$

which is equal to our right hand side of C_iC_{n-i-1}. Therefore, we have shown that $a(n,i+1)-a(n,i)=C_iC_{n-i-1}$. Now, we can plug this into our recurrence, getting

$$\sum_{i=0}^{n-1}a(n,i+1)-a(n,i),$$

which telescopes down to $a(n,n)-a(n,0)$. This simplifies to

$$=\frac{2n-n}{2n(n+1)}\binom{2n}{n}\binom{2n-2n}{n-n}-\frac{-n}{2n(n+1)}\binom{0}{0}\binom{2n}{n}$$

$$=\frac{1}{2(n+1)}\binom{2n}{n}+\frac{1}{2(n+1)}\binom{2n}{n}=\frac{1}{n+1}\binom{2n}{n}.$$

Thus, we are done; we have shown that $\sum_{i=0}^{n-1}C_iC_{n-i-1}=C_n$. ■

Theorem 1.3 (Second Recurrence Relation)**.** *The Catalan numbers also satisfy the recurrence* $C_{n+1}=\frac{4n+2}{n+2}C_n$*, with* $C_0=1$*.*

Although this recurrence may be less prominent than the first in the discussion of Catalan numbers, it's actually quite powerful, as we'll discover in the rest of this paper. Before we begin looking at applications of Theorem 1.3, let's prove it.

Proof of Theorem 1.3. Let's consider the ratio $\frac{C_{n+1}}{C_n}$. Using the explicit formula for C_n, we get

$$\frac{C_{n+1}}{C_n}=\frac{\frac{1}{n+2}\binom{2n+2}{n+1}}{\frac{1}{n+1}\binom{2n}{n}}=\frac{(n+1)(2n+2)!n!^2}{(n+2)(2n)!(n+1)!^2}=\frac{(2n+1)(2n+2)(n+1)}{(n+2)(n+1)^2}=\frac{4n+2}{n+2}.$$

Therefore, $C_{n+1}=\frac{4n+2}{n+2}C_n$. ■

This recurrence relation helps us gain greater insight on some properties of the Catalan numbers, such as growth rate and its generating function, as we'll discuss later in this paper.

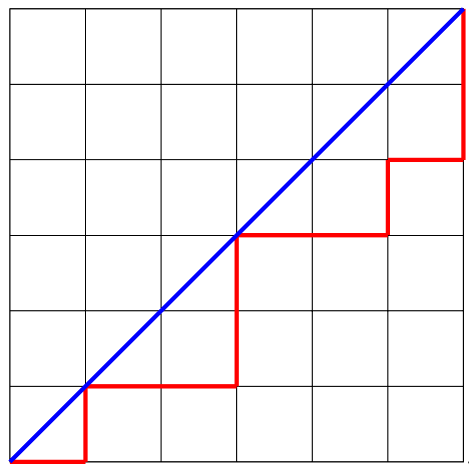


Figure 1. A 6×6 Dyck Path, denoted by $RURRUURRURUU$

2. Catalan Objects

Now that we have introduced the Catalan numbers algebraically, its time to move onto a discussion of the Catalan numbers in their natural habitat: combinatorics.

There are a few main ways to prove that the Catalan numbers answer a combinatorial problem. The first way is to show a bijection from the problem to another problem that Catalan numbers answer; if we show that two problems are equivalent, and we already know that the Catalan numbers answer one, then the Catalan numbers must answer the other.

The second way to show that Catalan numbers answer a combinatorial problem is to show that the answer to the problem adheres to either the Catalan numbers' initial conditions and recurrences, or the Catalan numbers' explicit formula. Although we will not review the proof in this paper, one classic proof that the Catalan numbers count the number of Dyck paths uses the second strategy, showing that the number of Dyck paths is equal to the Catalan numbers.

Definition 2.1 (Dyck paths)**.** An $n \times n$ Dyck path is a path from $(0, 0)$ to (n, n) taking steps to the north and east and never going above the line $y = x$ [RS21].

Theorem 2.2 (Dyck paths)**.** *Catalan numbers count the number of $n \times n$ Dyck paths.*

It is sometimes useful to denote Dyck paths as a sequence of Rs and Us, in which the Rs represent a movement rightward by one unit and the Us represent a movement upward by one unit. Figure 1 shows one example of a Dyck Path and the corresponding sequence of Rs and Us.

Now that we have defined the Dyck paths and established that the Catalan numbers count them, we can discuss other combinatorial questions the Catalan numbers answer. Specifically, we will investigate a simple proof that uses a bijection to prove that the Catalan numbers answer the problem.

Question 2.3 (Bertrand's Ballot Problem)**.** *Two candidates participate in an election in which Candidate A and Candidate B both receive n votes. What is the number of ways that the votes can be counted such that Candidate B is never ahead of Candidate A?*

Solution To Question 2.3. We can consider each way to count the votes as a sequence of As and Bs. An A in the sequence represents a vote being counted for Candidate A, and a B in the sequence represents a vote being counted for Candidate B. Thus, we can consider the sequences of length $2n$ containing n As and n Bs. One such sequence is $AAA \cdots AAABBB \cdots BBB$. We seek to find the number of ways to permute this sequence such that the number of As is always greater than or equal to the number of Bs. We notice that this restricted permutation problem sounds familiar; it mirrors the idea of Dyck paths, in which the number of steps to the right must always be greater than or equal to the number of steps upward (to ensure that the path never crosses $y = x$).

To formally show that the number of ways to count the vote is equal to the number of Dyck paths on an $n \times n$ board, we form a bijection. We define a function f from X, the set of valid sequences of As and Bs, to Y, the set of Dyck paths on an $n \times n$ board. Our function f works as follows: first, it converts the As and Bs in the sequence to Rs and Us respectively. Then, it forms a Dyck Path using the sequence of Rs and Us, as described above—letting an R in the sequence represent a movement one unit rightward and letting a U in the sequence represent a movement one unit upward. It is straightforward to show that f is a function and not just a relation, so we will move onto showing that f is a bijection.

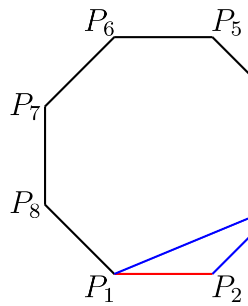
Now, to show that f is a bijection, we have to show that it is both an injection and a surjection. First, we show that f is an injection by proving that if $f(x) = f(y)$, then $x = y$. This is relatively simple; consider the identical Dyck paths $f(x)$ and $f(y)$. We can write out the sequences of Rs and Us that correspond to these Dyck paths, yielding sequence S_1 and S_2. We note that because the Dyck paths are identical, the sequences S_1 and S_2 must also be the same. Then, we replace the Rs with As and the Us with Bs. Clearly, our new sequences R_1 and R_2 are identical, since identical transformations acted on identical sequences. Therefore, we have shown that if $f(x) = f(y)$, then $x = y$, so we have an injection.

Now, we must show that f is a surjection. Thus, we must show that at least one sequence of As and Bs corresponds to each Dyck Path on an $n \times n$ board. We consider an arbitrary Dyck Path D with sequence representation S_3. We can recover a valid sequence of As and Bs by converting the Rs in S_3 to As and the Us to Bs, getting some sequence R_3. We claim that this sequence R_3 is a valid sequence of As and Bs for any Dyck Path D. Since the number of Rs is greater than or equal to the number of Us for any point in the sequence S_3, the number of As is greater than or equal to the number of Bs for any point in the sequence R_3. Therefore, we have recovered a valid x such that $f(x) = y$ for any y, and thus we have shown that f is a surjection.

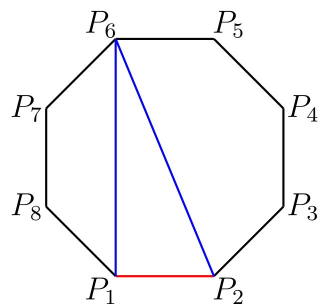
Now, we have shown that f is both an injection and a surjection. Therefore, $f : X \to Y$ is a bijection, and thus, $|X| = |Y|$. Since we know that $|Y| = \frac{1}{n+1}\binom{2n}{n}$, $|X| = \frac{1}{n+1}\binom{2n}{n} = C_n$. Thus, the number of ways that the votes can be counted such that Candidate B is never ahead of Candidate A is C_n. ■

Although formalizing the bijection proved slightly tedious, the bijection between Dyck paths and the ways to count the ballots was relatively clear to see. For our next theorem, we will consider a counting problem for which the bijection is less obvious.

Definition 2.4 (Triangulations of a Convex Polygon)**.** A triangulation of a convex polygon is formed by drawing diagonals between non-adjacent vertices, provided you never intersect another diagonal (except at a vertex), until all possible choices of diagonals have been used [Hea02].



(a) Triangle $P_1P_2P_3$ separates the octagon into the triangle $P_1P_2P_3$ and the heptagon $P_3P_4P_5P_6P_7P_8P_1$.



(b) Triangle $P_1P_2P_6$ separates the octagon into the triangle $P_1P_2P_6$, the quadrilateral $P_1P_6P_7P_8$, and the pentagon $P_2P_3P_4P_5P_6$.

Figure 2. An Octagon in two stages of the inductive step.

Now that we've defined triangulations, we can state our theorem.

Theorem 2.5 (Triangulations of a Convex Polygon)**.** *The number of triangulations of a convex n-gon is C_{n-2}.*

Where do we start in forming a bijection? Forming a bijection to Dyck paths seems difficult, as there doesn't appear to be a simple way to convert triangulations to Dyck paths. Many of the classic proofs that the Catalan numbers count the number of triangulations of a convex $(n+2)$-gon use a bijection to binary trees or complete parenthesizations. However, in the interest of trying something different and perhaps more insightful, we will attempt to use the first strategy we discussed: showing that the number of triangulations of a convex $(n+2)$-gon satisfies the same recurrence and has the same initial conditions as C_n.

Proof of Theorem 2.5. We will proceed by strong induction on n. Starting with $n = 3$ as our base case, we clearly see that there is only $1 = C_1$ way to triangulate a convex triangle. Now, we consider an n-gon with vertices $P_1, P_2, \ldots P_{n-1}, P_n$. We will consider the edge P_1P_2. We note that P_1P_2 must belong to exactly one triangle in the triangulation. Therefore, we can do casework on the point that completes the triangle with P_1 and P_2.

Consider the point P_3 completing the triangle. Since P_1, P_2, and P_3 are consecutive vertices on our n-gon, our triangle $P_1P_2P_3$ splits the n-gon into an $(n-1)$-gon and the triangle, as depicted in Figure 2a. Our inductive hypothesis tells us that there are C_{n-3} ways to triangulate the $(n-1)$-gon, and our original triangle is clearly already triangulated. Therefore, the case of the point P_3 contributes $C_{n-3} = C_{n-3}C_0$ triangulations to our total.

Next, we consider the point P_4 completing our triangle. We see that $P_1P_2P_4$ splits the n-gon into the triangle $P_1P_2P_4$, another triangle, and an $(n-2)$-gon. Again, by our inductive hypothesis, we know that there are C_1 ways to triangulate a 3-gon and C_{n-4} ways to triangulate an $(n-2)$-gon. Since the triangulations of the 3-gon and the $(n-2)$-gon are independent, we multiply to get a contribution of C_1C_{n-4} triangulations from the case of the point P_4.

Similarly, we find that the case of P_5 forms a 4-gon and an $(n-3)$-gon, giving us a contribution of C_2C_{n-5} cases from the case of the point P_5.

Generally, the triangle P_i splits the n-gon into an $(i-1)$-gon and an $(n-i+2)$-gon, as shown in Figure 2b. The $(i-1)$-gon and the $(n-i+2)$-gon together give us a contribution of $C_{i-3}C_{n-i}$ by our inductive hypothesis.

Therefore, summing over i, we get that the total number of triangulations is

$$\sum_{i=3}^{n} C_{i-3}C_{n-i} = \sum_{i=0}^{n-3} C_i C_{n-3-i} = C_{n-2}.$$

Therefore, we have completed our inductive step, as we have shown that if our hypothesis is true for all $i < n$, then our hypothesis is true for n. Thus, we are done, and we have proven that the number of triangulations of a convex $(n+2)$-gon is C_n. ■

There are other Catalan Objects we can consider; in fact, Richard Stanley has compiled over 200 combinatorial interpretations of the Catalan numbers [Sta15]! With the strategies described above, one is well-equipped to prove that the Catalan numbers answer many of the combinatorial equivalences described. Instead of attempt to chip away at the more than $\binom{200}{2}$ bijections between Catalan Objects one can prove, we will shift our discussion to analytic combinatorics.

3. Analytic Combinatorics on the Catalan numbers

We will begin our discussion of analytic combinatorics with the generating function for the Catalan numbers.

Theorem 3.1 (Generating Function)**.** *The generating function for C_n is $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.*

The classic proof of the Generating Function of the Catalan numbers uses the first recurrence relation: $\sum_{i=0}^{n-1} C_i C_{n-i-1}$. However, in my research, I did not come across any proof of the Generating Function using the less prominent second recurrence relation; thus, in the interest of trying something new (and perhaps less formulaic), we will take a different approach to the classic one, instead using the multiplicative recurrence relation for the Catalan numbers.

In our proof of Theorem 3.1, we will need to take the limit of $C(x)$ at $x = 0$. Therefore, we need to show that there is some nonzero radius of convergence around $x = 0$; if there isn't any nonzero radius of convergence around $x = 0$, then there are no values of x from which to approach $x = 0$ in our limit.

We do not need to be exact, as we can take the limit at $x = 0$ as long as there is some finite radius of convergence. Thus, we will just show that the radius of convergence is at least $\frac{1}{5}$.

Lemma 3.2. *The radius of convergence of $C(x) = \sum_{n=0}^{\infty} C_n x^n$ around $x = 0$ is at least $\frac{1}{5}$.*

Proof of Lemma 3.2. To show that $C(x)$ has a radius of convergence of at least $\frac{1}{5}$ around $x = 0$, we must show that $C(x)$ converges for all $x : -\frac{1}{5} \leq x \leq \frac{1}{5}$.

Using our recurrence relation $C_{n+1} = \frac{4n+2}{n+2} C_n$, we note that $C_{n+1} < 4C_n$. Thus, $C_n < 4^n C_0 = 4^n$. Although we can make stricter bounds for C_n, we find that this is sufficient, as we have

$$\sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^n 4^n = \frac{1}{1-\frac{4}{5}} = 5.$$

Since $C_n < 4^n$, $\frac{C_n}{5^n} < \frac{4^n}{5^n}$. Thus,

$$\sum_{i=0}^{\infty}\left(\frac{1}{5}\right)^n C_n < \sum_{i=0}^{\infty}\left(\frac{1}{5}\right)^n C_n = 5.$$

Therefore, $C(\frac{1}{5})$ converges by the comparison test. We note that if $0 < i < \frac{1}{5}$, then $C(i) < C(\frac{1}{5})$, so $C(i)$ converges by the comparison test. Therefore, we have shown that if $0 < x \leq \frac{1}{5}$, then $C(x)$ converges.

Now, we must show that if $-\frac{1}{5} \leq x < 0$, then $C(x)$ converges. We can use the comparison test again, this time comparing $C(-i)$ with $C(i)$ for $0 < i \leq \frac{1}{5}$. Denote the nth terms of $C(-i)$ and $C(i)$ as a_n and b_n respectively. We note that $a_{2n} = b_{2n} = i^{2n}C_{2n}$ and $a_{2n+1} = -i^{2n+1}C_{2n+1} < b_{2n+1} = i^{2n+1}C_{2n+1}$. Since $a_n \leq b_n$ for all nonnegative n,

$$C(-i) = \sum_{j=0}^{\infty} a_j < \sum_{j=0}^{\infty} b_j = C(i).$$

Since we have already shown that the right hand side converges for $0 < i \leq \frac{1}{5}$, we know that the left hand side, $C(-i)$, also converges.

Therefore, we have shown that if $-\frac{1}{5} \leq i < 0$ or $0 < i \leq \frac{1}{5}$, then $C(i)$ converges (and $C(0)$ converges trivially). Thus, we have shown that $C(x)$ has a radius of convergence of at least $\frac{1}{5}$, and we are done. ■

Now that we have shown that $C(x)$ has a finite radius of convergence around $x = 0$, we can prove Theorem 3.1.

Proof of Theorem 3.1. We seek to find $C(x)$ by manipulating our second recurrence relation: $C_{n+1} = \frac{4n+2}{n+2}C_n$. We multiply by $n+2$ on both sides of our recurrence, getting $(n+2)C_{n+1} = (4n+2)C_n$. Now, we multiply both sides by x_n and sum over n, getting

$$\sum_{n=0}^{\infty}(n+2)C_{n+1}x^n = \sum_{n=0}^{\infty}(4n+2)C_n x^n.$$

Distributing the $n+2$ and the $4n+2$ yields

$$\sum_{n=0}^{\infty} nC_{n+1}x^n + 2\sum_{n=0}^{\infty} C_{n+1}x^n = 4\sum_{n=0}^{\infty} nC_n x^n + 2\sum_{n=0}^{\infty} C_n x^n.$$

We now investigate each term individually, attempting to manipulate it to be in terms of $C(x)$. We will begin with the simplest one, the second term on the right hand side. We see that

$$2\sum_{n=0}^{\infty} C_n x^n = \boxed{2C(x)}.$$

Now, we will move onto the second term on the left hand side, which is similar. We can multiply by $\frac{x}{x}$ to get the summation in the form of $C_i x^i$

$$2\sum_{n=0}^{\infty} C_{n+1}x^n = \frac{2}{x}\sum_{n=0}^{\infty} C_{n+1}x^{n+1}.$$

Next, we reindex the summation to get

$$= \frac{2}{x}\sum_{n=1}^{\infty} C_n x^n = \frac{2}{x}(-1 + \sum_{n=0}^{\infty} C_n x^n) = \boxed{\frac{2}{x}\left(C(x) - 1\right)}.$$

Now, we move onto the first term on the right hand side. This term is more complicated because of the n, so we take inspiration from the fact that $\frac{d}{dx}x^n = nx^{n-1}$. We seek to eliminate the n, so using derivatives, we attempt to get rid of the factor of n in the summation. Using this strategy, we have

$$4\sum_{n=0}^{\infty} nC_n x^n = 4x\sum_{n=0}^{\infty} nC_n x^{n-1} = 4x\sum_{n=0}^{\infty} \frac{d}{dx}\left(C_n x^n\right) = 4x\frac{d}{dx}\left(\sum_{n=0}^{\infty} C_n x^n\right).$$

This is simply $\boxed{4xC'(x)}$.

Finally, we move onto the first term of the left hand side, which is the trickiest to get in terms of $C(x)$. We will first get rid of the factor of n, as above.

$$\sum_{n=0}^{\infty} nC_{n+1}x^n = x\sum_{n=0}^{\infty} nC_{n+1}x^{n-1} = x\sum_{n=0}^{\infty} \frac{d}{dx}(C_{n+1}x^n) = x\frac{d}{dx}\left(\sum_{n=0}^{\infty} C_{n+1}x^n\right).$$

We already know that $\sum_{n=0}^{\infty} C_{n+1}x^n = \frac{2}{x}(C(x)-1)$ from above, so we can plug that in to get

$$= x\frac{d}{dx}\left(\frac{2}{x}(C(x)-1)\right) = x\frac{d}{dx}\left(\frac{2}{x}C(x) - \frac{2}{x}\right).$$

Using the product rule for derivatives, we get

$$= x\left(-\frac{1}{x^2}C(x) + \frac{1}{x}C'(x) + \frac{1}{x^2}\right) = \boxed{\frac{-C(x)}{x} + C'(x) + \frac{1}{x}}.$$

Now, we can substitute our manipulated expressions back into our equation, getting

$$\left(\frac{-C(x)}{x} + C'(x) + \frac{1}{x}\right) + \frac{2}{x}\left(C(x)-1\right) = 4xC'(x) + 2C(x).$$

We now have a differential equation. Isolating $C'(x)$ gives us

$$C'(x) = \frac{(1-2x)C(x)-1}{x(4x-1)}.$$

The solution to the differential equation is

$$C(x) = \frac{c_1\sqrt{1-4x}}{x} + \frac{1}{2x} = \frac{1+c\sqrt{1-4x}}{2x}.$$

,

where the constant $c = 2c_1$. To find c, we examine $C(0)$. Because $C(x) = \sum_{n=0}^{\infty} C_n x^n$, $C(0) = C_0 = 1$. Additionally, since $C(x)$ is a polynomial function, and is thus continuous,

$$\lim_{x\to 0} C(x) = C(0) = 1;$$

recall that this limit is well-defined due to Lemma 3.2.

Therefore, we have that

$$\lim_{x\to 0} C(x) = \lim_{x\to 0} \frac{1+c\sqrt{1-4x}}{2x} = 1.$$

Plugging in $x = 0$ to $C(x)$ gives us $\frac{1+c}{0}$. There are two cases: $1+c \neq 0$ or $1+c = 0$. If $1+c \neq 0$, then $C(x)$ has an asymptote at $x = 0$. However, this isn't possible, as $C(x)$ is the sum of several continuous functions; thus, it must be continuous. Therefore, $1+c$ must

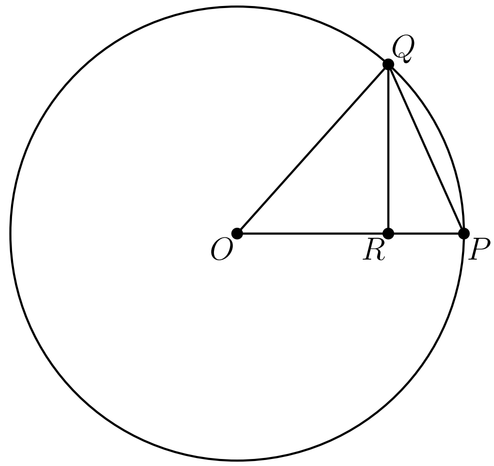


Figure 3. Diagram not to scale.

equal 0. Since that means that direct substitution evaluates to $\frac{0}{0}$, we can use L'hôpital's Rule to simplify our limit, getting

$$\lim_{x\to 0}\frac{1-\sqrt{1-4x}}{2x}=\lim_{x\to 0}\frac{4/2(1-4x)^{-1/2}}{2}=\lim_{x\to 0}\frac{1}{\sqrt{1-4x}}=1.$$

Therefore, we have confirmed that $c=-1$, and we have found the generating function for the Catalan numbers:

$$C(x)=\frac{1-\sqrt{1-4x}}{2x}.$$

■

Now that we've found the generating function for the Catalan numbers, let's put it to use.

Question 3.3. *Consider a circle centered at O with radius r. Points P and Q are on the circumference of circle O such that the altitude of $\triangle OPQ$ from Q to R has length* 1, *as depicted in Figure 3. Why do the Catalan numbers appear in the decimal expansion of RP, when r is of the form $5\cdot 10^n$?*

Solution to Question 3.3. Let's solve for RP. We know that $RP=OP-OR$, and since $OP=r$, we just have to find OR. Since $\triangle ORP$ is a right triangle, we find that $OR=\sqrt{OQ^2-QR^2}=\sqrt{r^2-1}$. Therefore, $RP=r-\sqrt{r^2-1}$. We seek to show that this is related to the generating function in some way. We will attempt to manipulate our expression to get it in the form of $\frac{1-\sqrt{1-4x}}{2x}$. We have

$$r-\sqrt{r^2-1}=\frac{1-\sqrt{1-\frac{1}{r^2}}}{1/r}.$$

Multiplying by $\frac{2r}{2r}$, we get

$$=\frac{2r}{2r}\left(\frac{1-\sqrt{1-\frac{1}{r^2}}}{1/r}\right)=\frac{1}{2r}\left(\frac{1-\sqrt{1-\frac{1}{r^2}}}{1/2r^2}\right).$$

By the definition of $C(x)$, this is equal to

$$= \frac{1}{2r}C(\frac{1}{4r^2}).$$

Now, we can plug in $r = 5 \cdot 10^n$. This yields

$$= \frac{1}{10^{n+1}}C\left(\frac{1}{10^{2n+2}}\right) = \frac{1}{10^{n+1}}\sum_{i=0}^{\infty}\frac{C_n}{10^{2nx+2x}}.$$

Therefore, our generating function reveals why the Catalan numbers appear in the decimal expansion of $r - \sqrt{r^2 - 1}$ when r is 5 times a power of 10.

■

Now that we've completed our discussion of the generating function for the Catalan numbers, we can begin unraveling the growth rate of Catalan numbers. First, we'll consider the ratio between consecutive elements, then we'll discuss an asymptotic approximation of the Catalan numbers.

Theorem 3.4. *As n grows, $\frac{C_{n+1}}{C_n}$ approaches* 4.

The theorem is fairly easy to prove since we already have a recursion for C_n that deals with the ratio between consecutive terms.

Proof of Theorem 3.4. We have that $C_{n+1} = \frac{4n+2}{n+2}C_n$. Therefore, $\frac{C_{n+1}}{C_n} = \frac{4n+2}{n+2}$. Finally,

$$\lim_{n\to\infty}\frac{4n+2}{n+2} = 4.$$

Thus, we are done. ■

This result leaves us with an interesting question. We know that the ratio between consecutive Catalan numbers approximates 4 for large n. Then, how much does 4^n differ from C_n? It turns out that Stirling's approximation can help us answer this question.

Theorem 3.5 (Stirling's approximation)**.** *Stirling's approximation states that*

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1.$$

Because the derivations of Stirling's approximation require prerequisite understanding not assumed in this paper, we will not prove it here [Con16].

Now that we have Stirling's approximation, we can begin discussing the asymptotic approximation for the Catalan numbers.

Theorem 3.6. *We can approximate Catalan numbers with the following limit:*

$$\lim_{n\to\infty}\frac{C_n}{4^n n^{-\frac{3}{2}}\pi^{-\frac{1}{2}}} = 1.$$

This theorem has some similar elements to Stirling's approximation. Additionally, the fact that C_n can be expressed in terms of factorials makes it even more enticing to use Stirling's approximation to attempt to prove Theorem 3.6, as we'll do in the following proof.

First Proof of Theorem 3.6. We will directly consider the limit of $\frac{4^n}{C_n}$ as n goes to ∞. This gives us

$$\lim_{n\to\infty}\frac{4^n}{C_n}=\lim_{n\to\infty}\frac{4^n(n+1)(n!)^2}{(2n)!},$$

by the explicit formula for the Catalan numbers. We now use Stirling's approximation, getting that our limit is equal to

$$\lim_{n\to\infty}\frac{4^n(n+1)(\sqrt{2\pi n}(n/e)^n)^2}{\sqrt{4\pi n}(2n/e)^{2n}}.$$

Simplifying the numerators and denominators yields

$$\lim_{n\to\infty}\frac{4^n(n+1)(2\pi n)(n/e)^{2n}}{2\sqrt{\pi n}(n/e)^{2n}2^{2n}}=\lim_{n\to\infty}\frac{4^n(n+1)\sqrt{\pi n}}{4^n}=\lim_{n\to\infty}(n+1)\sqrt{\pi n}\approx\lim_{n\to\infty}\sqrt{\pi}n^{\frac{3}{2}}.$$

Therefore, we have that

$$\lim_{n\to\infty}\frac{4^n}{C^n}=\lim_{n\to\infty}\sqrt{\pi}n^{\frac{3}{2}},$$

and thus

$$\lim_{n\to\infty}\frac{C_n}{4^n n^{-\frac{3}{2}}\pi^{-\frac{1}{2}}}=1.$$

■

Although we have only shown that the ratio is 1 as n tends to ∞, we can note that the ratio is fairly close to 1 even for smaller n; for $n = 20$, the ratio is approximately 0.946, and for $n = 50$, the ratio is approximately 0.978. The reason for this is that Stirling's approximation for $n!$ is also a good approximation for small n; since our result depends on Stirling's approximation, the rate of convergence transfers from the independent to the dependent approximation.

Although our proof is succinct because it uses Stirling's approximation, it may feel a bit uninspired, as we are just plugging in our approximation and simplifying. Therefore, we'll discuss another proof that may be more intuitive.

Before beginning the proof, let's state and prove a useful lemma.

Lemma 3.7.

$$\prod_{i=0}^{n-1}\left(1+\frac{3}{2i+1}\right)=\frac{\sqrt{\pi}\Gamma(n+2)}{\Gamma(n+\frac{1}{2})},$$

where the Gamma function $\Gamma(n)$ is an extension of the factorial function.

Proof of Lemma 3.7. We will prove our lemma by induction. Our base case is $n = 1$, which gives us

$$\prod_{i=0}^{0}(1+\frac{3}{2i+1})=1+3=4 \text{ and } \frac{\sqrt{\pi}\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}=\frac{2\sqrt{\pi}}{\sqrt{\pi}/2}=4.$$

Therefore, our base case is complete.

Now, we move onto our inductive step. We have that

$$\prod_{i=0}^{n-1}\left(1+\frac{3}{2i+1}\right)=\frac{\sqrt{\pi}\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}$$

by our inductive hypothesis. Now, we multiply both sides of the equation by $1+\frac{3}{2n+1}$ to get

$$=\prod_{i=0}^{n}\left(1+\frac{3}{2i+1}\right)=\frac{\sqrt{\pi}\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}\cdot\frac{2n+4}{2n+1}=\frac{\sqrt{\pi}\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}\cdot\frac{n+2}{n+\frac{1}{2}}.$$

Using the fact that $\Gamma(x+1)=x\Gamma(x)$, we get that this is equal to

$$=\frac{\sqrt{\pi}\Gamma(n+3)}{\Gamma(n+\frac{3}{2})},$$

so we have

$$=\prod_{i=0}^{n}\left(1+\frac{3}{2i+1}\right)=\frac{\sqrt{\pi}\Gamma(n+3)}{\Gamma(n+\frac{3}{2})},$$

and thus our inductive step is done. ■

Now that we have proven Lemma 3.7, we are ready to complete our second proof of Theorem 3.6.

Second Proof of Theorem 3.6. We begin with our recurrence relation for C_n: $C_{n+1}=C_n\frac{4n+2}{n+2}$, with initial value $C_0=1$. Clearly, we have

$$C_n=\prod_{i=0}^{n-1}\frac{4i+2}{i+2}=\prod_{i=0}^{n-1}4-\frac{6}{i+2}.$$

We know that $4^n=\prod_{i=0}^{n-1}4$, so we have

$$\frac{4^n}{C_n}=\prod_{i=0}^{n-1}\frac{4}{4-\frac{6}{i+2}}=\prod_{i=0}^{n-1}\frac{1}{1-\frac{3}{2i+4}}=\prod_{i=0}^{n-1}\frac{2i+4}{2i+1}=\prod_{i=0}^{n-1}\left(1+\frac{3}{2i+1}\right).$$

By Lemma 3.7, we know that

$$\prod_{i=0}^{n-1}\left(1+\frac{3}{2i+1}\right)=\frac{\sqrt{\pi}\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}.$$

We have a few options here. We can use Stirling's approximation as before, or we can attempt to evaluate the ratio without it using the properties of the Gamma function. Since the second derivative of $\Gamma(x)$ is positive for nonnegative x (which follows from $\Gamma(x+1)=x\Gamma(x)$), $\Gamma(x)$ increases at a faster rate for larger x. Thus, we can set bounds on $\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}$ as follows:

$$\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-1)}\cdot\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}<\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}\cdot\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}<\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}\cdot\frac{\Gamma(n+\frac{7}{2})}{\Gamma(n+2)}.$$

Simplifying, then taking the square root, we get

$$\sqrt{\frac{\Gamma(n+2)}{\Gamma(n-1)}}<\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}<\sqrt{\frac{\Gamma(n+\frac{7}{2})}{\Gamma(n+\frac{1}{2})}}.$$

Using $\Gamma(x+1)=x\Gamma(x)$, we can simplify the bounds to get

$$\sqrt{n^3-n}<\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})}<\sqrt{n^3+4.5n^2+5.75n+1.875}.$$

Dividing by $n^{\frac{3}{2}}$ on both sides, we get

$$\sqrt{1-\frac{1}{n^2}} < \frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})n^{\frac{3}{2}}} < \sqrt{1+\frac{4.5}{n}+\frac{5.75}{n^2}+\frac{1.875}{n^3}}.$$

Since

$$\lim_{n\to\infty}\sqrt{1-\frac{1}{n^2}} = \lim_{n\to\infty}\sqrt{1+\frac{4.5}{n}+\frac{5.75}{n^2}+\frac{1.875}{n^3}} = 1,$$

we know that $\lim_{n\to\infty}\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})n^{\frac{3}{2}}} = 1$, and thus

$$\lim_{n\to\infty}\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})} = n^{\frac{3}{2}}.$$

Since we've already established that

$$\frac{4^n}{C_n} = \frac{\sqrt{\pi}\Gamma(n+2)}{\Gamma(n+\frac{1}{2}},$$

we now have

$$\lim_{n\to\infty}\frac{4^n}{C_n} = \lim_{n\to\infty}\frac{\sqrt{\pi}\Gamma(n+2)}{\Gamma(n+\frac{1}{2}} = \lim_{n\to\infty}\sqrt{\pi}n^{\frac{3}{2}}.$$

Rearranging, we get

$$\lim_{n\to\infty}\frac{C_n}{4^n n^{-\frac{3}{2}}\pi^{-\frac{1}{2}}} = 1,$$

and we are done.

■

Thus, we have proven Theorem 3.7 without using Stirling's approximation, instead just using our multiplicative recurrence and the properties of the Gamma function.

Our discussion of the growth rate of the Catalan numbers prompts one final question: Why do Dyck paths, Triangulations of Convex Polygons, or other Catalan Objects see an increase of a factor of approximately 4 as n grows? We can dive into casework for Dyck paths (or any other Catalan Object), but it's messy. Furthermore, checking small cases doesn't seem promising, as the ratio between consecutive Catalan numbers isn't close to 4 for small n.

It seems that our only hope is finding why the number of Dyck paths grows by $\frac{4n+2}{n+2}$, but this also seems rather difficult, especially since $\frac{4n+2}{n+2}$ isn't even an integer for most n. Therefore, we close our discussion of Catalan numbers with this question remaining.

References

[Con16] Keith Conrad. Stirling's formula, 2016.
[Dav06] Tom Davis. Catalan numbers. 2006.
[Hea02] Danny Heap, Nov 2002.
[HN99] Ferran Hurtado and Marc Noy. Graph of triangulations of a convex polygon and tree of triangulations. *Computational Geometry*, 13(3):179–188, 1999.
[KS06] Thomas Koshy and Mohammad Salmassi. Parity and primality of catalan numbers. *The College Mathematics Journal*, 37(1):52, 2006.
[RS21] Simon Rubinstein-Salzedo. *Combinatorics.* 2021.
[Sta15] Richard P Stanley. *Catalan numbers.* Cambridge University Press, 2015.

[Wei02] Eric W. Weisstein. Gamma function. 2002.