Ramsey Theory: Extensions, and Applications

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Contents

1 Abstract

In this paper, we introduce Ramsey Theory and how it is related to graph theory, discuss some important extensions and results of Ramsey's Theory, and consider some applications of Ramsey Theory in other fields of math as well as in the real world.

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2 Introduction

Throughout this paper, we will be talking in great detail about graphs, so we introduce some terms and notation which will be used throughout this paper.

For our purposes, **graphs** are defined as multiple nodes connected by edges. As a note, throughout this paper, the words **vertex** and **node** will be used interchangeably. Consider for example a graph G. A **subgraph** $g \in G$ is a smaller section or part of a graph G. A **clique** is a connected subgraph of G. We say G is a **complete graph** if every node is connected to every other node, and let K_n denote the complete graph with n nodes.

We will be coloring different edges of these graphs, so for an edge between two nodes v_1 and v_2, let $C(v_1, v_2)$ denote the color of that edge. Also, for every node $v \in G$, let $d_b(v)$ and $d_r(v)$ denote the number of blue and red edges emanating from node v, respectively. A given combination of d_b's and d_r's for a graph is one possible **coloring** of that graph, and we assume that in a coloring, every edge is colored exactly one color.

2.1 Ramsey's Theorem

We now introduce the centerpiece of this paper:

Theorem 1 (Ramsey's Theorem)**.** *Given two positive integers m and n such that $m, n \geq 2$, there is a minimum positive integer, denoted by $R(m, n)$, such that in any red-blue coloring of $K_{R(m,n)}$, one can find at least one blue clique on m vertices or a red clique on n vertices.*

However, this isn't such an obvious fact. Let's see why such a minimum number exists for all pairs (m, n). The proof in this paper was an elegant proof presented by [Mat].

Proof. We prove this by induction on m and n. Firstly, we prove some properties of $R(m, n)$ that will be important in this proof:

1. $R(m, n) = R(n, m)$. This is true by symmetry because switching red and blue edges makes no difference.

2. $R(2, n) = R(n, 2) = n$. For our purposes, we will focus on proving $R(n, 2) = n$, then $R(2, n) = n$ follows by Property 1 mentioned above. First, we show that n vertices are sufficient: if we have at least one

red edge, we have a red clique of size 2, as desired. Now, consider the case where none of the edges are red, i.e. all of them are blue. Then, we have a blue clique on n vertices, and we are done. Notice that if we have less than n vertices and all of the edges are blue, then we will neither have a red clique of size 2 or a blue clique of size n. Thus, we need at least n vertices.

To show that $R(m, n)$ is well defined for all pairs of positive integers (m, n) such that $m, n \geq 2$, we must show that such a number is finite. Now, we begin our induction.

Base Case: $m2$. Then we have $R(n, 2) = R(2, n) = n$, which is a finite number. Hence, it is well defined.

Inductive Step: Assume that $R(m-1, n)$ and $R(m, n-1)$ are both well defined. We will show that

$$R(m, n) \leq R(m-1, n) + R(m, n-1),$$

which implies that $R(m, n)$ is well defined. To prove this, consider one special vertex v. Then, we have $d_b(v) + d_r(v) + 1 = R(m-1, n) + R(m, n-1)$. Thus, we have two possibilities: either $d_b(v) \geq R(m-1, n)$ and $d_r(v) < R(m, n-1)$, or $d_r(v) \geq R(m, n-1)$ and $d_b(v) < R(m-1, n)$.

If $d_b(v) \geq R(m-1, n)$, then consider the subgraph created by those $d_b(v)$ vertices: we can guarantee a blue clique of size $m-1$ or a red clique of size n. Since v is connected to each of the vertices of this subgraph by a blue edge, by adding on v to this subgraph, we can guarantee a blue clique of size m or a red clique of size n.

A similar argument applies if $d_r(v) \geq R(m, n-1)$. This completes the proof. $\square$

3 Important Results and Extensions

3.1 The Infinite extension of Ramsey's Theorem

Ramsey's Theorem can in fact be extended to infinite graphs.

Theorem 2 (Infinite Ramsey Theorem)**.** *Consider an infinitely large graph G that is colored by a finite number of colors. Then there exists an infinitely large clique $\mathcal{C} \subseteq G$ such that all of the edges connecting the vertices of $\mathcal{C}$ have the same color.*

We present a nice, clean, and easy proof of the theorem that uses a clever inductive construction approach as in [Mis].

Proof. Let V_0 be the set of vertices in G. Consider one of these vertices $v_0 \in V_0$. By the Pigeonhole Principle, since we have an infinite number of edges coming from v_0 but only a finite number of colors, at least one of these colors must be coloring an infinite number of the edges coming from v_0. Let one of those such colors be denoted c_0. Now, let V_1 denote the vertices of G such that for all $v' \in v_1$, we have $C(v_0, v') = c_0$. Note that $V_1 \subset V_0$.

We can do the same process again on V_1, since it is also an infinite set of vertices: consider one vertex $v_1 \in V_1$, then by the Pigeonhole Principle, there must be at least one color, say color c_1, such that there are an infinite number of vertices v'' connected to v_1 such that $C(v_1, v'') = c_1$. Let the set of vertices connected to v_1 by an edge of color c_1 be denoted as V_2. We can repeatedly apply this construction to generate the sets $V_0, V_1, V_2, V_3, \ldots$, where we are considering one vertex $v_i \in V_i$ and then generating the infinite set V_{i+1} from all of the vertices connected to v_i by an edge of color c_i.

Now, we make some observations: for all $i \geq 0$,

1. $v_i \in V_i$,
2. $V_{i+1} \subset V_i$, and
3. $C(v_i, v) = c_i$ for all $v \in V_{i+1}$.

Now, we prove the following lemma:

Lemma 3.1. *For any integers i, j such that $0 \leq i < j$, it is true that*

$$C(v_i, v_j) = c_i.$$

Proof. By property 1, we have $v_j \in V_j$. Then, by property 2, we have $V_j \subset V_{j-1} \subseteq \cdots \subseteq V_{i+1}$, so $v_j \in V_{i+1}$. Finally, by property 3, we have $C(v_i, v_j) = c_i$.

Revisiting the Pigeonhole Principle, since we have finitely many colors but infinitely many edges, at least one color, let's say c, occurs infinitely many times in G. Let our clique $\mathcal{C}$ have the set of vertices V where $V = \{v_i : i \geq 0 \text{ and } c_i = c\}$. Then we claim $\mathcal{C}$ is *the* clique we are looking for: $\mathcal{C}$ is infinite, and for any two vertices $v_i, v_j \in V$, by Lemma 4.1 we have $C(v_i, v_j) = c_i = c$. Thus, G has an infinite monochromatic clique $\mathcal{C}$. $\square$

It is possible to prove Ramsey's Theorem from the infinite version of the theorem using a proof by contradiction in conjunction with an infinite construction idea similar to the one above. We leave this as an exercise to the reader.

3.2 Ramsey Numbers

Ramsey numbers, which are all numbers $R(m, n)$ over all pairs of positive integers (m, n) (where $m, n \geq 2$), have been an area of much research for the past 100 years. Specifically, mathematicians have been trying to improve the bounds on Ramsey numbers. We know the exact values for very few Ramsey numbers because computing Ramsey numbers for large m and n is very difficult; in fact, we don't even know the exact value of $R(5, 5)$! All we know is that $43 \leq R(5, 5) \leq 48$.

So, instead of focusing on exact values, mathematicians have changed their focus to proving stronger bounds for diagonal Ramsey numbers - numbers in the form $R(k, k)$. All of the bounds presented in this section are of diagonal Ramsey Numbers.

In 1935, Paul Erdős and George Szekeres proved that

$$R(m+1, n+1) \leq \binom{m+n}{m}.$$

This was a big deal; Erdős was regarded as one of the best mathematicians of his time, so his work in Ramsey Theory brought the world's attention to this relatively young field. In fact, Erdős is commonly regarded as the person who made Ramsey Theory a popular subjet.

This bound was also the best mathematicians had for almost 50 years until 1980 when Vojtěch Rödl proved that

$$R(m+1, n+1) \leq \frac{\binom{m+n}{m}}{c \log^c(m+n)}$$

for some constant $c > 0$.

In 1988, Robert Thomason beat this bound by showing that

$$R(m+1, n+1) \leq \binom{m+n}{m} m^{-n/(2m)+c/\sqrt{\log k}}$$

for some constant $c > 0$ as long as $m \geq n$.

Then, in 2006, a new breakthrough by David Conlon showed that

$$R(m+1, m+1) \leq m^{-c \log m / \log \log m} \binom{2m}{m}.$$

Thomason's and Conlon's bounds were remarkable. However, very recently in 2020, MIT student Ashwin Sah made a ground-breaking improvement in the upper bound of Ramsey Numbers [Sah20] and showed that for all $k \geq 3$, there is a constant $c > 0$ such that

$$R(m+1, m+1) \leq e^{-c(\log m)^2} \binom{2m}{m}.$$

Research on Ramsey Numbers is still continuing today, and new discoveries are being made in the field every day. However, even though today we have much more computing power and mathematical knowledge to tackle Ramsey Theory, it is fun to invent hypothetical situations which place the fate of humanity in the hands of not superheroes, but mathematicians.

One such interesting hypothetical situation is known as the "Alien Invasion Problem", which is centered around a famous quote by Paul Erdős:

> "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack." [Lam16]

This quote from Erdős reminded the international scientific community about how little we know about Ramsey Numbers despite all of the progress we have made; if we were asked to find $R(6, 6)$, it would be impossible. We would have to look through almost 10^{1550} possible graphs (for reference, there are between 10^{78} and 10^{82} atoms in the observable universe).

4 Applications

4.1 Ramsey Theory in General

Ramsey theory, in general, has to do with finding order in a substructure of a larger structure of a given size. For example, finding monochromatic cliques within a larger graph. The first paper classified as a part of Ramsey Theory was published in 1927 by Bartel Leendert van der Waerden, a Dutch mathematician. He is famously accredited with Van der Waerden's Theorem (theorem statement from [Ste15]):

Theorem 3 (Van der Waerden's Theorem)**.** *For any two given positive integers s and p, there is a minimum number n such that for any coloring of $\{1, 2, 3, \ldots, n\}$ with p colors, there is an arithmetic sequence of s numbers where all of those numbers have the same color. We denote n as $W(p, s)$.*

Van der Waerden numbers are also very challenging to calculate, and we only know a few of them. For example, $W(2, 3) = 9, W(3, 4) = 293$. They grow very fast as well; in fact, $W(5, 8) > 493, 700$! (values taken from [BCT18].)

Another interesting theorem early in Ramsey Theory is known as Schur's Theorem, proposed by Issai Schur:

Theorem 4 (Schur's Theorem)**.** *If the set of natural numbers $\mathbb{N}$ is colored by a finite number of colors, then there exist natural numbers a, b, c such that $a + b = c$ and a, b, c all have the same color. For a given $n \in \mathbb{N}$, let $S(n)$ denote the smallest natural number such that if $S(n)$ is colored by n colors, then there exist $a, b, c \in S(n)$ such that $a + b = c$ and a, b, c all have the same color.*

Resembling the nature of Ramsey Theory, Schur Numbers also are very hard to calculate, and we know very few. For example, $s_1 = 1, s_2 = 4,_3 = 13, s_4 = 44$, but we don't know the exact value of s_5; all we know is that $s_5 \geq 160$ (values taken from [Bou15]).

Finally, we present one more theorem that was considered one of the first theorems in Ramsey Theory, Richard Rado's Theorem (as stated in [HTB20]):

Theorem 5 (Rado's Theorem)**.** *Let A be an $1 \times m$ matrix, and let $\boldsymbol{x}$ be a $m \times 1$ matrix of variables such that $A\boldsymbol{x}$ is a system of linear equations. Then, if we color the solution space of this system of equations with n colors (where $n \in \mathbb{N}$), then there is a monochromatic set of solutions if and only if the columns of A can be partitioned into $C_1 \cup C_2 \cup \cdots \cup C_k$ such that*

1. $\sum\limits_{i=0}^{k} C_i = \vec{0}$.

2. *for all $j > 1$, every element in C_j can be expressed as a linear combination of elements in $C_1 \cup C_2 \cup \cdots \cup C_{j-1}$.*

4.2 Ramsey Theory in Real life

Ramsey Theory is not only prevalent in other fields of math but also in many real world scenarios, some of which I present here.

4.2.1 War and Peace

One famous historical example of Ramsey Theory appearing in real life is when the English scholar Sir Woodson Kneading observed that from 600 to 400 BCE, there were 42 instances where five powers in a peaceful region went to war when a sixth power entered the region. He said:

> "I noticed that either (1) three, four, or five of them formed an alliance and, thinking themselves quite powerful, merged armies and attacked the other lords, or (2) there were three or more of them who were pairwise enemies, and in that case war broke out among the factions..." [BI16]

This essentially was the observation that $R(3,3) = 6$: either 3 were allies or 3 were pairwise enemies. The fact that this observation was made long before Ramsey Theory was established is very striking.

4.2.2 The Birthday Party

The Birthday Party [Gas] is a classic application of Ramsey Theory and is usually the setup used to introduce the theorem to beginners.

Question 4.1 (The Birthday Party Question)**.** *How many people would we need at a Birthday Party to guarantee that there is a group of* 3 *people all of whom either know each other or do not know each other?*

The answer is $R(3,3) = 6$. This problem is closely related to Ramsey Theory and introduces the topic very nicely.

However, Ramsey Theory enables us to ask not only about 3 people, but maybe 4 people. For example...

Question 4.2 (The Birthday Party Question)**.** *How many people would we need at a Birthday Party to guarantee that there is a group of* 4 *people all of whom either know each other or do not know each other?*

As it turns out, the answer to this question is 18. We will not be going into the details of why this is, but a great resource that goes more into this is [BI16].

4.2.3 Electricity Pricing

Haiming Li and Jia He published a paper titled "An Application of the Ramsey Number in the Electricity Pricing" [LH16] which focused on electricity pricing in China. They studied the applications of Ramsey Theory to sales price cross-subsidies by investigating the relationship between "revenues and costs attributable to a single commodity or commodity combinations." Especially considering the reforms going on in the electricity pricing industry during the time, countries around the world had high concerns for sales price cross-subsidies. They applied the mathematical model of Ramsey Pricing. However, they concluded that more research needed to be done on how new problems could arise from the solution to the Ramsey pricing model in different stages of development of China's electric power industry, for example how it would affect low-income residents.

5 Final Remarks

In this paper, Today, we talked about Ramsey's Theorem and discussed the applications of Ramsey Theory. Finding order in substructures is a vast topic, and as a result, Ramsey Theory encompasses a large part of mathematics. Despite the amount of research that has already been done on Ramsey Theory and Ramsey Numbers, we still have much to learn. Much research is still underway in this young and active field. For example, mathematicians are studying variants of Ramsey Numbers, such as Induced, Size, and Generalized Ramsey Numbers [CFS15].

In the future, we can hope to discover new bounds on Ramsey Numbers and possibly new theorems in this field.

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