Pattern-Avoiding Permutations

Amandip Dutta

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Abstract

In this essay, we will introduce and discuss the basis of permutations and Wilf-equivalence. We will then explain simple, 3-length, permutations, and how to find permutations that are 132-avoiding, as well as a special 231-avoiding algorithm proposed by Knuth.

1 Introduction

The term permutation refers to the mathematical technique used to determine the number of possible arrangements in a given set where the order of the arrangement matters. The common generalized formula is expressed as shown below:

$$P(a, b) = \frac{a!}{(a-b)!}$$

known that

- a is the number of total elements in the set
- b is the number of selected elements that are arranged in a specific order

Permutations are also an arrangement of entries into a sequence. What makes permutations so versatile is that they are used in almost all forms of mathematics and science. For example there is a permutation known as a **stack-sortable permutation** which is a permutation whose entries can be sorted by a certain algorithm. These are also known as 231-avoiding permutations, which will be discussed below.

Given the basic definition and usages of permutations, pattern avoidance in permutations counts the specific arrangements in a set that match or avoid a particular pattern. For example, given two linear arrays (permutations) $\alpha = \alpha_1, \alpha_2, \alpha_3, ...\alpha_x$ and $\beta = \beta_1, \beta_2, \beta_3, ...\beta_x$, we can make conclusions based on the smaller occurrences (smaller permutations) of the whole permutation. If array α have the same order of a specific number sequence (ex: 123) as array β, then we could say that α contains β. If array α did not have the same order of a specific number sequence as array β we can say α avoids β. We can keep track of the order in which the integers appear, so we can highlight all the possible arrangements. A numerical example of this is 5243 would be an arrangement of 4132 in the permutation 654321.

Another way we can show whether a permutation contains or avoids another permutation is if one permutation has a subsequence that can reduce down to the other permutation. Considering the previous example above, we can say α contains β if α has a subsequence that reduces to β and vice-versa. Another important theory in pattern avoidance are the reverses and complements of a permutation. As the name suggests, given a permutation

$$a = a_1, a_2, ...a_n$$

the reverse of it is

$$a^r = a_n, a_n - 1, ...a_1$$

The complement of a permutation can be made by replacing the nth smallest number of a permutation a with the nth largest number from permutation a. To easier illustrate how a reverse and complement of a permutation come from a standard permutation, we can use a dot diagram to illustrate this. The numbers in a permutation represent heights relative to each other. For example, a dot diagram for the 24351 is shown down below:

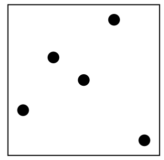

Figure 1.1: The dot diagram for 24531.

Given this dot diagram, we can draw the complement and reverse of this permutation. The reverse would be drawn as if the permutation 24531 flipped over the vertical-axis (y-axis). The complement of the permutation would be flipped over the horizontal-axis (x-axis). The reverse and complement are shown down below, respectively:

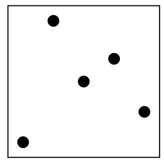

Figure 1.2: The reverse of 24531.

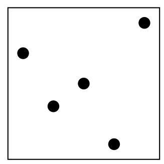

Figure 1.3: The complement for 24531.

2 Wilf-Classes and Wilf-Equivalence

Given two permutations π and σ, they could be said to be Wilf-equivalent if and only if they share an equivalence relation with one another. The equivalence relation includes any permutation that is reflexive, symmetric, or transitive of another permutation. Because there are many different ways two permutations can be Wilf-classes of one another, there are special things we can notice with low sized permutations. For example, permutations $123, 231$ and 132 are all Wilf-equivalent. Research in this field has proven that any two permutations of size 3 are Wilf-equivalent. c-Wilf equivalent permutations is when two permutations attain the same generating function. Thus, we can say:

Definition 2.1. Two permutations π, σ are c-Wilf equivalent if

$$P_(\pi)(0,a) = P_(\sigma)(0.a)$$

The smallest example of a c-Wilf equivalence that does not arise from 123 symmetries is given by $1342 \overset{s}{\sim} 1432$. Two permutations can be strong c-Wilf equivalent when they both start with the same number, aside from zero. This brings us to our next definition:

Definition 2.2. Two permutations π, σ are strong c-Wilf equivalent if

$$P_\pi(n,a) = P_\sigma(n,a)$$

It is obvious that a strong c-Wilf equivalence implies c-Wilf Equivalence. Another easy way to figure out whether a c-Wilf equivalent is strong, is if the permutation is non-overlapping. Take the example below:

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Example 2.1. Given two permutations $a = a_1, a_2, ...a_n$ and $b = b_1, b_2, ...b_n$. If permutation a is not overlapping with permutation b, then given $a \sim b$, we can say $a \overset{s}{\sim} b$ Also, given that the permutations are in standard form, we can say:

Definition 2.3 Let a and b be permutations that are standard and non-overlapping. If $a \sim b$, then $a_1 = b_1$ and $a_n = b_n$. Despite Definition 2.3 applying to only non-overlapping permutations, we can have a similar definition without the restriction of having a non-overlapping permutation. AS mentioned above, for non-overlapping patterns, c-Wilf equivalence is the same as strong c-Wilf equivalence, so therefore we can say:

Definition 2.4 Let a and b be permutations that are standard. If $a \overset{s}{\sim} b$, then $a_1 = b_1$ and $a_n = b_n$.

An example of standard, strongly c-Wilf-equivalent pairs are the permutations $a = 123546$ and $b = 124536$.

3 123 and 132 Avoiding Permutations

Before we begin elaborating on specific 123 or 132 avoiding permutations, let us go over an example.

Assume that there are n people standing in a line. How many lineups exist, such that every person each person can see everyone who is shorter than them and precede them in the line? To solve this problem, we will have to enumerate permutations of $[n]$. Assume that from 1 to n people, 1 is the shortest and n is the tallest, and $n = 6$. One lineup that would not work is 142653. It would not work because 2 would not be able to see 1, since 4 is blocking them. One example that would work is 563241. The most efficient way to figure out which lineups would work is by noticing the sub-permutations in the bigger permutation. For example, if we reduced n to 3, how many different lineups could there be now? 123 and 132 would work, and 321 would obviously not work because the people would only be seeing people taller than him. Using this, we can assume that this "123" and "132" permutation would also appear in larger permutations. However, 123 would only appear in a single permutation (ex: if $n = 8$, we can only say 12345678). On the other hand, 132 would appear in most permutations, like the example of 563241 as shown above when $n = 6$. This pattern of the smallest appearing first, the largest appearing second, and the middle number appearing last is what we can call a [132] permutation. Using this, we can write a definition regarding 132 patterns:

Definition 3.1 Let x, y, and z be three numbers in a permutation that appear respectively. If $x < z < y$, then we cam say that the entries x, y, and z form a 132-pattern.

Similarly, we can create a similar definition to this regarding 123 patterns:

Definition 3.2 Let a, b, and c be three numbers in a permutation that appear respectively. If $a < b < c$, then we cam say that the entries a, b, and c form a 123-pattern.

Certain permutations that don't contain the pattern 132, can be stated as follows:

Definition 3.3 Given a permutation p, if there are no entries in p that form 132-patterns, then p is called 132-avoiding.

Using this, we can find out how many permutations of length n are 132-avoiding. Suppose we have a 132-avoiding permutation called a, where entry n is in the ith position. From this, we can claim that any entry to the left of n must be larger than any numbers to the right of n. To be able to see this, let us prove this by contradiction, such that there is an entry x and an entry y on the right of n such that x is less than y. Then, the entries x, n, and y form a 132-pattern, which is therefore a contradiction. From this, we can say entries 1,2, ... $n - i$ are on the right of n, and entries $n - i + 1$, $n - i + 2$, ... $n - 1$ are on the left of n. The $i - 1$ entries on the left of n must form a 132-avoiding permutation, which they can do in $f(i - 1)$ ways. We can do the same thing on the other side, in which we can get $f(n - i)$ ways. From this we can say:

Definition 3.4 There are exactly $f(i - 1)f(n - i)$ 132-avoiding n permutations in which n is the ith position.

To create a recurrence relation from this, assuming that $f(0) = 1$, we can say:

$$f(n) = \sum_{i=1}^{n} f(i-1)(n-i) \tag{1}$$

Therefore, from this we can conclude how to find how many 132 avoiding permutation exists such that entry n is in the ith position.

4 231 Avoiding Permutations and the Catalan Numbers

231-avoiding permutations are similar to 132 and 123-avoiding permutations in the way that the entries appear in the permutation: first entry is middle number, second entry is the largest number, and the last entry is the smallest number. From this we can again write a definition:

Definition 4.1 Let d, e, and f be three numbers in a permutation that appear respectively. If $f < d < e$, then we cam say that the entries d, e, and f form a 231-pattern.

231-avoiding permutations originally came from the problem of a sorting an input sequence using a stack. This was first proposed by Knuth in 1968, and gave the following linear time algorithm:

1. Initialize an empty stack.
2. For each input value x:
   (a) While the stack is not empty and such that x is larger that the top item on the stack, pop the stack to the output.
   (b) Push x onto the stack.
3. While the stack is nonempty, pop it to the output.

Knuth realized that this algorithm correctly sorts a couple input sequences, but doesn't sort all of them. For example, the permutation 321 is sorted correctly because they are popped in the order: 123. However, the permutation 231 is sorted incorrectly because the algorithm first pushes 2 and pops it once it sees 3, which causes 2 to be output before 1, instead of after it. Because of this, this algorithm can be considered to be a 231-avoiding algorithm.

What makes this permutation even more interesting is that it doesn't depend on the values of the permutation, instead it focuses on the relative order of the largest and smallest numbers. Knuth observed that if the algorithm fails to sort an input, then the input cannot be sorted with a single stack.

The pops and pushes that were demonstrated by Knuth's algorithm sorts a stack-sortable permutation. He reinterpreted it as a push as a left parenthesis, and a pop as a right parenthesis. Every Dyck string comes from a permutation that is stack-sortable, therefore every two two different stack-sortable permutations produce different Dyck strings. Because of this, the number of stack-sortable permutations of length n is the same as the number of Dyck strings with the length 2n, which is also known as the Catalan number:

$$C_n = \tfrac{1}{n+1}\binom{2n}{n}$$

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