Pattern Avoidance in Permutations

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First, we will go over some key definitions relating to Pattern Avoidance. We will discuss the properties of 3-digit patterns, including their relation with the Catalan numbers, the bijection between different 3-digit patterns, etc. Then, we will move on to 4-digit patterns, first by showing that there are more 1325 avoiding permutations than 1234 avoiding permutations for any permutation with more than 7 elements, then we will show the Wilf Equivalency between certain patterns. In the end, we will go over the approximation for the number of permutations that avoid certain patterns and the Stanley Wilf conjecture.

1 3 Digit Patterns and Wilf Equivalency

Definition 1.1: Let p be a permutation on n, and let $q = q_1q_2q_3 \cdots q_k$ be a permutation on k, where $n \geq k$. If we choose k elements from p while preserving their order and label them $a_1a_2a_3a_4 \cdots a_k$, and if for every i, j, if $q_i < q_j$ and $a_i < a_j$, then we say that the elements $a_1a_2a_3 \cdots a_k$ forms a *q-pattern*.

Example: Let $p = 124356$ and $q = 123$. Then the elements $1, 3, 6$ in p forms a *q-pattern* pattern.

Definition 1.2: Let p be a permutation on n, and let $q = q_1q_2q_3 \cdots q_k$ be a permutation on k, where $n \geq k$. If no k entries of p forms a *q-pattern* , then p is a *q-avoiding* permutation

Example: Let $p = 34512$, $q = 132$, $r = 231$, then p avoids q but does not avoid r.

The number of *q-avoiding* n-permutations is denoted by $S_n(q)$. It is very tempting to ask how many q avoiding permutations of length n are there. Fortunately, there is an easy way to calculate $S_n(q)$ when q is a 3 digit permutation.

Definition 1.3: For two permutations q and r, q and r are said to be *Wilf-Equivalent* if $S_n(q) = S_n(r)$ for all n.

Example: The pattern 123 is *Wilf-Equivalent* with the pattern 321, because 321 is the reverse of 123 and the reverse of every permutation that avoids 123 avoids 321. This sets up a natural bijection between 123-avoiding permutations and 321-avoiding permutations. Further more, we claim that:

Lemma 1.4: All 3-digit patterns are *Wilf-Equivalent.*

Proof. There are 6 possible 3-digit patterns: 123, 132, 231, 213, 312, 321. Any pattern is *Wilf-Equivalent* with it's reverse, so we only need to show that $S_n(123) = S_n(132) = S_n(312)$. We define the *complement* of an n-permutation $p = p_1p_2p_3p_4 \cdots p_n$, p_c to be $p_{c1}p_{c2}p_{c3} \cdots p_{cn}$ where $p_{ci} = n+1-p_i$. For example, the complement of 21345 is 45321. Moreover, 132 is the complement of 312. Observe that if a permutation p contains the pattern 132, then p_c contains 312(We will formally define and prove reversal and complementation later in part 2). Thus, $S_n(132) = S_n(312)$.

Now we want to show that $S_n(123) = S_n(132)$. Define the *left-to-right minimum* to be an entry such that it is smaller than all entries before is. For instance, the left-to right minima in 43521 are $4, 3, 2, 1$. Note that the first entry and 1 are always a left-to-right minimum. Now we construct a bijective function f from the set of all 123-avoiding n-avoiding permutations to all 132-avoiding n-permutations. It is defined as follows: take any 123-avoiding n-permutation, we fix all the left-to-right minima, remove all the other entries, and then place them back from left to right such that every entry is the smallest element that is still larger than the previous left-to-right minimum. For example, $f(465132) = 456123$. Thus, the resulting permutation is composed of many increasing sequences and any of the elements in any given sequence is larger than any elements in the sequences that succeed it.Obviously, such sequence is 132 avoiding.

The inverse of f is described as such: WE fix all the left-to-right minima of p, and then put all the other elements into the empty slots between them in decreasing order. For example, $f^{-1}(456231) = 465231$. Thus, all the left-to-right minima are preserved. Note that both the left-to-right minima and the remaining entries form a decreasing sequence. Therefore, we obtain a permutation that is a union of two decreasing sequences. Such permutation has to be 123-avoiding, as for every 3 entries chosen, at least 2 of them belong to the same sequence. This completes the proof. □

To find $S_n(q)$ for all q where the length of q is 3, we only need to find $S_n(132)$.

Corollary 1.5: $S_n(132) = C_n = \frac{1}{2n+1}\binom{2n+1}{n}$, where C_n is the Catalan number.

Proof. Suppose p is a n-permutation that avoids 132, and the element n is in the i'th entry. The smallest elements before p_i has to be greater than the largest element that proceeds p_i. To prove this, lets assume the contrary: there is an element x before p_i and element y after p_i such that $x < y$. If that's the case, then the sequence x, n, y forms a 132 pattern, which is impossible. The sequences formed by the entries $p_1p_2p_3p_4 \cdots p_{i-1}$ and $p_{i+1}p_{i+2}p_{i+3}p_{i+4} \cdots p_n$ both have to be 132-avoiding, and since n can be in any position, we obtain the following recurrence:

$$S_m(132) = \sum_{i=1}^{n} S_{i-1}(132)S_{n-i}(132)$$

Which is the same recurrence as the Catalan numbers. Since their initial conditions are also the same, they have to be the same. □

Thus, we obtain the formula for the number of n-permutations that avoids q, where q is any 3-digit pattern:

Theorem 1.6: For any permutation pattern of length q, we have

$$S_n(q) = C_n = \frac{1}{2n+1}\binom{2n+1}{n}$$

We went over single pattern avoidance, we can take it a step further and count the number of n-permutations that avoid two different 3-digit patterns, denoted by $A_n(q, r)$.

Lemma 1.7:

(a) $A_n(123, 132) = A_n(123, 213) = A_n(231, 321) = A_n(312, 321)$;

(b) $A_n(132, 213) = A_n(231, 312)$;

(c) $A_n(132, 231) = A_n(213, 312)$;

(d) $A_n(132, 312) = A_n(213, 231)$;

(e) $A_n(132, 321) = A_n(123, 231) = A_n(123, 312) = A_n(213, 321)$

Proof. All of the identities above can be derived through the reversal and complementation technique demonstrated earlier. □

Thus, we split all double restriction permutations into five cases, proposition 1.8 through 1.12 will deal with them separately.

Proposition 1.8: $A_n(123, 132) = 2^{n-1}$ for all $n \geq 1$

Proof. Let σ be a n permutation that avoids 123 and 132. If $\sigma_n = n$ then everything else must be in decreasing order. Else, if $\sigma_k = n$ for some $1 \leq k < n$, then for all i's where $i < k$, we have $\sigma_i > n - k$, and $\sigma_1 > \sigma_2 > \sigma_3 \cdots \sigma_{k-1}$, while $(\sigma_{k+1}\sigma_{k+2} \cdots \sigma_n)$ also forms a $(123, 132)$ avoiding sequence. Thus we obtain the recurrence

$$A_n(123, 132) = \sum_{k=1}^{n-1} 1 + A_{n-k}(123, 132)$$

It follows from the recurrence that

$$A_n(123, 132) = 2^{n-1}$$

□

Proposition 1.9: $A_n(132, 213) = 2^{n-1}$ for all $n \geq 1$

Proof. Let σ be such a permutation. If $\sigma_n = n$, then σ is the identity permutation. Else, if $\sigma_k = n$ for some $1 \leq k < n$, then we must have $\sigma_i = n - k + i$ for $1 \leq i \leq k$, and $(\sigma_{k+1}\sigma_{k+2} \cdots \sigma_n)$ also avoids (132, 213). Since $A_n(132, 213)$ satisfy the same initial value as $A_n(123, 132)$, they must be the same. □

Proposition 1.10: $A_n(132, 231) = 2^{n-1}$ for all $n \geq 1$

Proof. Let σ be such a permutation, we can create two $(132, 231)$-avoiding permutations of length $n + 1$ by either inserting the element $n + 1$ at the beginning or the end. Thus, we have

$$A_n(132, 231) = 2A_{n-1}(132, 231).$$

Plugging in the initial conditions, we have

$$A_n(132, 231) = 2^{n-1}$$

□

Proposition 1.11: $A_n(132, 312) = 2^{n-1}$ for all $n \geq 1$

Proof. Let σ be such a permutation. If $\sigma_1 = n$ then $\sigma_i = n + 1 - i$ for all $1 \leq i \leq n$. Otherwise, if $\sigma_k = n$ where $2 \leq k \leq n$, then $(\sigma_1\sigma_2 \cdots \sigma_{k-1})$ must be a $(132, 312)$ avoiding permutation while $\sigma_{k+i} = n - k - i + 1$ for all $1 \leq i \leq n - k$. Thus,

$$\begin{aligned} A_n(132, 312) &= 1 + \sum_{k=2}^{n} A_{k-1}(132, 312) \\ &= 1 + \sum_{k=1}^{n-1} A_k(132, 312) \end{aligned}$$

Which is the same one in the previous propositions. Plugging in the initial condition and solving the recursion gives:

$$A_n(132, 312) = 2^{n-1}$$

□

Proposition 1.12: $A_n(132, 321) = \binom{n}{2} + 1$ for all $n \geq 1$

Proof. Let σ be such a permutation. If $\sigma_n = n$ then all the entries that precedes n form a permutation that avoids $(132, 321)$. Otherwise, n can be in any position σ_k where $1 \leq k \leq n-1$, and $(\sigma_1\sigma_2 \cdots \sigma_{k-1})$ and $(\sigma_{k+1}\sigma_{k+2} \cdots \sigma_n)$ form increasing sequences and consist of the largest $k - 1$ terms and the smallest $n - k$ terms, respectively. Thus,

$$A_n(132, 321) = A_{n-1}(132, 321) + n - 1$$

Solving the recursion completes the proof □

2 4 Digit Patterns

As we've shown, it is relatively easy to find the number of q-avoiding permutations if the length of q is three. We now shift our focus to q-avoiding permutations, if the length of q is four. There are 24 possible explicit q-patterns of length 4, but we can easily reduce the amount of patterns that must be given explicit attention. Further reduction requires the use of new tools to identify Wilf-equivalence.

Definition 2.1: A permutation $p = p_1 p_2 \cdots p_m$ has a reverse p^r, defined as $p^r = p_m p_{m-1} \cdots p_1$.

Example: For $q = 2143$, $q^r = 3412$.

Definition 2.2: A permutation $p = p_1 p_2 \cdots p_m$ has a *complement* p^c, where the ith position is occupied by $m + 1 - p_i$.

Example: For $p = 1243$, $p^c = 4312$.

It is natural to ask about the relationships between a permutation or pattern p, its reverse p^r, and its complement p^c.

Proposition 2.1 For all positive integers n,

$$S_n(p) = S_n(p^c) = S_n(p^r)$$

Proof. Suppose that we have a pattern q that is contained by an arbitrary sequence $p = p_1 p_2 p_3 \cdots p_n$. If we take the reverse of this sequence, then it is clearly evident that q^r is contained by p^r. Since reverses are unique for each permutation, we have established a bijection between finite sets, so they must be equal in size. This leaves the same number of permutations of $[n]$ that avoid q^r, so they must be the same in size. Similarly, if we take the complement of p, we obtain the sequence $(n + 1 - p_1) \cdots (n + 1 - p_n)$. We can manipulate a relation among two arbitrary elements of pattern q, q_1, q_2 to prove that q^c is contained by p^c.

$$\begin{gathered} q_1 < q_2 \\ -q_1 > -q_2 \\ n + 1 - q_1 > n + 1 - q_2 \end{gathered}$$

Therefore, all the relations in a permutation are flipped by taking complements. Since the set of relations is flipped in both q^c and p^c, it follows that $S_n(q) = S_n(q^c)$. $\square$

It is also natural to question the Wilf-equivalence of inverse permutations and patterns. We define inverses with the normal combinatorial definition.

Theorem 2.1 For all positive integer n and patterns q,

$$S_n(q) = S_n(q^{-1})$$

Theorem 2.2 Suppose k is a positive integer, and p is a permutation of $[k+1, \ldots, k+r]$. It follows that

$$S_n(12\ldots kp) = S_n(k(k-1)\ldots 1p)$$

By taking various reverses, complements, and applying theorems (2.1) and (2.2), we can limit ourselves to only 4 permutations of interest,

$$1234, 1324, 1342, 2413.$$

However, there is one more Wilf-equivalence among this set of permutations.

Theorem 2.3 For all positive integer n,

$$S_n(1342) = S_n(2413).$$

Thus, we can reduce the study of all 24 explicit q-patterns of length 4 to the study of only three *non-Wilf-equivalent* patterns,

$$1234, 1342, 1324.$$

There are many fascinating properties about these three patterns. Some of the most interesting include (surprising) closed forms regarding $S_n(q)$, where q is one of the three q-patterns of interest.

Theorem 2.4 For all n, we find that

$$S_n(1342) = (-1)^{n-1}\frac{7n^2-3n-2}{2} + 3\sum_{i=2}^{n}(-1)^{n-i}\cdot 2^{i+1}\cdot\frac{(2i-4)!}{i!(i-2)!}\cdot\binom{n-i+2}{2}$$

This may be proven using generating functions by deriving the rational form of the ordinary generating function for $S_n(1342, x)$. The explicit formula is the first of its kind for a length greater than 3, which are counted by the Catalan numbers C_n. A similar *exact* formula exists for $S_n(1234)$.

Theorem 2.5 For all n,

$$S_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}$$

No equivalent exact enumeration exists for $S_n(1324)$. There does, however, exist a recurrence derived by Marinov[1].

Other interesting facts about the q-patterns of length 4 when considered in terms of inequalities. Upon viewing successive values of $S_n(1234)$ and $S_n(1324)$, it is evident that $S_n(1234)$ is generally less than or equal to $S_n(1324)$.

Proposition 2.2 For all n,

$$S_n(1234) < S_n(1324)$$

Proof. To prove this argument, we will need to classify permutations.

Definition 2.3 Permutations p and q are in the same class if the left ot right minima and right to left maxima of both p and q are the same and are in the same positions.

Example: 51234 and 51324 are in the same class.

We first prove that each class contains only one 1234 avoiding permutation. By fixing the positions of minima and maxima, placing elements into the remaining positions creates an avoiding permutation. If this contained a 1234 subsequence, two entries would necessarily be identical, so this cannot occur. Then, every class must necessarily contain at least one 1234 avoiding permutation. In fact, this is the only 1234 avoiding permutation. If any two entries are placed in increasing order (i.e. changing from the original decreasing arrangement), they would necessarily form a 1234 pattern.

We now prove that each class contains at least one 1324 avoiding permutation. Finding a pattern of this sort involves choosing a pattern so that its first element is a left to right minimum and the fourth a right to left maximum. Therefore, proving that a permutation avoids 1324 is equivalent to showing that it doesn't contain 1324 with a local minimum as the first element and a local maximum as the last element.

Any 1324 containing permutation will contain a pattern of this type. By interchanging the second and third elements of this pattern, we can create a new permutation within the same class without changing the minima or maxima. After repeating this process for at most $\binom{n}{2}$ times, the resulting permutation will not have a 1324 pattern. Thus, every class contains at least one of these permutations. It remains to prove that for all $n \geq 7$, classes that contain more than one 1324 avoiding permutation exist. The class $3-1-7-5$ for n = 7 will contain two permutations that contain 1324; we can maintain this structure by simply adding newer elements for $n > 7$ to the beginning or end of the permutation. The new class will obtain two 1324 avoiding permutations. Thus, since each class contains at most 1 permutation that avoids 1234, and each class contains at *least* one permutation that avoids 1324, we may conclude that for all $n \geq 7$,

$$S_n(1234) < S_n(1324).$$

□

3 Stanley Wilf Conjecture

Here, we show a proof for the Stanley-Wilf Conjecture[2]. Namely:

Theorem 3.1 Recall that $S_n(\sigma)$ is the set of permutations of $[n]$ that avoid the pattern σ. Then, there exists some $c = c(\sigma)$ such that $S_n(\sigma) \leq c^n$ for every n.

In order to prove the conjecture, we begin with a proof of the *Furedi - Hajnal Conjecture*. We then show that this theorem necessarily implies the result of the Stanley - Wilf Conjecture,

3.1 Furedi - Hajnal Conjecture

Definition 3.2 Let A and P be matrices where every entry is either a 0 or a 1. We say that A contains P if P is a submatrix of A. That is, we are allowed to delete rows and columns of A to form P, but we are not allowed to permute them in any way. If A does not contain P, we say that A *avoids* P.

Definition 3.3 Define $F(n, P)$ to be the maximum number of 1 entries in a nxn matrix avoiding P.

Theorem 3.4 $F(n, P) = O(n)$.

The crux idea is to formulate a recurrence relation. We assume P to be a kxk permutation matrix and A to be a nxn matrix with $F(n, P)$ 1-entries that avoid P. Furthermore, we assume that k^2 divides n.

Let $S_{i,j}$ be a square submatrix of A consisting of entries $a_{i',j'}$, where $i' \in [k^2(i-1), k^2 i]$, and $j' \in [k^2(j-1), k^2(j)]$. We further define B with entries $b_{i,j}$ to be the submatrix of A which $b_{i,j} = 0$ whenever all the entries of $S_{i,j}$ are 0.

A block is *wide* whenever it has 1-entries in at least k different columns, and it is *tall* whenever it has 1-entries in at least k different rows.

Lemma 3.5 B avoids P.
Clearly, if B didn't avoid P, then the block of k 1-entries of B matches with any arbitrary 1-entry from the corresponding block of A, meaning that A doesn't avoid P, which is a contradiction.

Lemma 3.6 The number of wide blocks of the form $C_j = \{S_{i,j} | i = 1, 2, \ldots, n/k^2\} < k\binom{k^2}{k}$.

We can just formulate a scheme that over counts the number of wide blocks. By pigeonhole principle, there exists k blocks in C_j which have a 1-entry in the same column. Now, we can just pick k of the k^2 corresponding 1-entries in the columns. However, these entries of A do not avoid P, so $|C_j| < k\binom{k^2}{k}$. Similarly, we have:

Lemma 3.7 The number of wide blocks of the form $R_i = \{S_{i,j} | j = 1, 2, \ldots, n/k^2\} < k\binom{k^2}{k}$.

Lemma 3.8 $f(n, P) \leq (k-1)^2 f(n/k^2, P) + 2k^3\binom{k^2}{k}n$.

Let $|X_1|$ be the set of tall blocks, $|X_2$ the set of wide blocks, and $|X_3|$ the set of blocks that are neither wide nor tall. Clearly, any wide or tall block contains at most k^4 1-entries. A block in $|X^3|$ contains at most $(k-1)$ 1-entries. So, our recursion is $f(n, P) < k^4|X_1| + (k-1)^2|X_3|$, and applying Lemma 3.7, and 3.8, we have our result.

Proof of Theorem 3.4 We can prove a slightly stronger statement. That is, we show that $f(n, P) \leq 2k^4\binom{k^2}{k}n = O(n)$.

This can be found by strong induction and an application of Lemma 3.8.

3.2 Deduction of Stanley-Wilf

Definition 3.9 Let $T_n(P)$ be the set of nxn matrices that avoid P.

An important corollary of this statement is that $S_n(\sigma) \leq |T_n(P)|$. Note that a permutation σ avoids another permutation π if and only if the permutation matrix σ avoids the permutation matrix π. Now, suppose P is a permutation matrix corresponding to some σ, then $T_n(P)$ contains the permutation matrices of all permutations that avoid σ, so $S_n(\sigma) \leq |T_n(P)|$.

At this point, we can try and find a bound of $T_n(P)$, which also places a bound on $S_n(\sigma)$.

Theorem 3.10 For any permutation matrix P, there exists a constant $c = c_P$ such that $|T_n(P)| \leq c^n$.

Proof of Theorem 3.10

Just note $|T_{2n}(P)| \leq |T_n(P)|15^{f(n,P)}$. We can find this by partitioning T_{2n} into 2x2 blocks, and then setting that 2x2 block to 0 if all the entries are 0, and setting it equal to 1 otherwise. From Lemma 3.5, this new block B also avoids P. Any matrix B is a mapping from at most 15^w matrices of $T_{2n}(P)$, where w is the number of 1-entries in B. Since $w < f(n,P)$, the result follows.

References

[1] R. R. D. Marinov, “Counting 1324-avoiding permutations,” *The Electronic Journal of Combinatorics*, 2003.

[2] G. T. Adam Marcus, “Excluded permutation matrices and the stanley–wilf conjecture,” 2003.