An Introduction to Ramsey Theory

Aathmanathan Muruganathan

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Abstract

In this expository paper we explore the fundamentals of Ramsey Theory, named after Frank Ramsey, a pioneer in the field. Virtually, Ramsey Theory entails finding order in a substructure of a structure of given size. We start this paper with several graph theory fundamentals, and then move on to explore the Friends and Strangers problem, which we extend to a graph theoretical perspective. Finally, we explore the proof behind Ramsey's Theorem.

1 Graph Theory Preliminaries

We assume some level of competence with graph theory, but we will provide a couple of definitions that will be useful later on. This section will be terse, as to move on to the main material.

Definition 1.1. A **graph** is a structure consisting of two sets V and E, where each element on V is a vertex and each element of E is an edge. The set of vertices of a graph G is $V(G)$. Similarly, the set of edges of G is $E(G)$.

Graphs are usually represented by pictures, as we have done so in Figure 1. Vertices are depicted as circle or dots, and each edge is a line connecting two vertices.

Definition 1.2. An edge which connects the same vertex to itself is a **loop**. A set consisting of two or more edges of a graph, if sharing the same ends, are called a set of **multiple edges**. [1]

Definition 1.3. A **simple** graph is one which has no loops or multiple edges. A **multigraph** is a graph that has at least one loop or multiple edge.

Refer to Figure 2 to see an example of a simple graph and a multigraph.

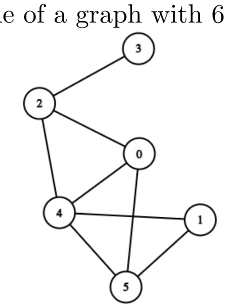
Definition 1.4. The **order** of a graph, denoted $|V|$. is the number of vertices it has, and the **size** of a graph, denoted $|E|$ is the number of edges it contains.

Definition 1.5. Given u, $v \in V$, if $uv \in E$, then u and v are **adjacent**. If an edge e has a vertex v as an end, then e and and v are **incident**

Definition 1.6. The set of vertices adjacent to v is the **neighborhood** of v.

Definition 1.7. The degree of a vertex v is the number of edges incident with v, and is denoted $\deg(v)$. [1]

Figure 1: An example of a graph with 6 vertices and 8 edges.



Definition 1.8. The **complete graph** on n vertices, K_n, is the graph of order n where $uv \in E$ for all $u,v \in V$.

We will now present some definitions regarding **subgraphs**, which will be highly useful in our proof of Ramsey's Theorem.

Definition 1.9. A graph H is a **subgraph** of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. [1]

Note that any graph is a subgraph of itself. This is not true for **proper subgraphs**.

Definition 1.10. A subgraph is knows as a **proper subgraph** if its vertex and edge sets are not the same as the vertex and edge sets of its "parent graph".

Definition 1.11. A subgraph H of G is an **induced subgraph** of G if H contains the same edges as G between its own vertices.

Lastly, we will define a **clique**.

Definition 1.12. A **clique** of G is a complete subgraph of G. The **clique number** of G is denoted $\omega(G)$ and is equal to the largest order of the largest complete graph that is a subgraph of G. [1]

2 An Interesting Problem

In this section, we explore a famous problem in Ramsey Theory, the **Friends and Strangers** problem. The Friends and Strangers problem asks to following: In a party with $k \geq$ 3people, each person is either a friend or a stranger to another

Figure 2: Example of a simple graph (on left) and a multigraph (on right).

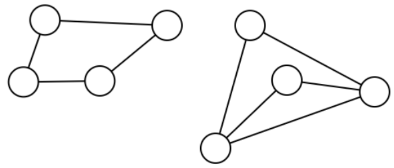
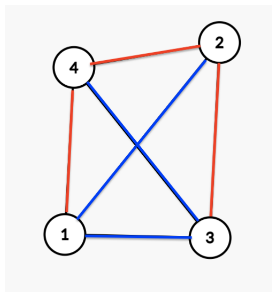


Figure 3: An example representation of the Friends and Strangers problem. Note that 1 and 2, 3 and 4, along with 1 and 3 are pairwise friends, while 1 and 4, 2 and 4, along with 2 and 3 are pairwise strangers. This disproves the $k = 4$ case.



person (friendship is mutual, as well as being a stranger to someone else). What is the smallest value of k such that three people are all (pairwise) strangers or are all (pairwise) friends?

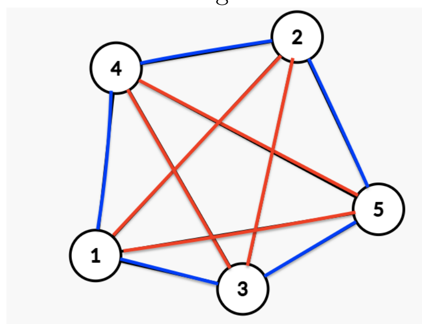
At first glance this may seem to be a problem that can easily be calculated by hand, and this is indeed true. The tricky part, though, is to prove that the answer obtained is the smallest one possible.

To make thinking about this problem easier, we can draw a diagram with each person represented by a dot, a blue line representing friendship, and a red line representing strangers. Such a diagram is shown in figure 3.

For the sake of example, let us look at the case k=3. Call the people at the party person A, person B, and person C. We easily find that k=3 does not work, since we can let person A be friends with both B and C, and we can let B and C be strangers to each other. We can also easily see that k=4 doesn't work; look at figure 3.

After going through a few more examples, you will probably find that k=6 is the least possible value. You can easily show that k=5 doesn't work (see figure 4) and we have already shown that 3 and 4 don't work. We will prove that k=6 is the smallest value possible by contradiction. Denote the six people people A, B, C, D, E, and F respectively. Assume that no three are friends and no three are strangers. Choose one person, say, person E, to analyze. We now have two cases to look at: Person E has less than 3 (0 to 2) friends, and person E has 3 or more (3 to 5) friends. Let's start with the second case. Since E has at least

Figure 4: An example showing that $k = 5$ does not work. Indeed, there is no group of three pairwise friends or strangers.



3 friends, we can assume, without loss of generality, that the relationship show in in Figure 5.A holds. [2] Since no 3 people can be mutual friends, C and F must be strangers. Likewise, B and C must be strangers. If B and F are friends, our condition that there are no three people who are friends breaks. If B and F are strangers, our condition that no three people are stranger breaks. Thus, we have a contradiction for this case. Now let us turn to case 1. We can take the same diagram we used for case one, and switch all the original friendships to stranger relationships. See figure 5.B to see what this new relationships looks like. We use a similar argument as we did with case 1 — B and C, as well as C and F, must be friends so that no three people are all strangers. B and F cannot be friends, since then there will be three mutual friends, and B and F cannot be strangers, because that will cause three people to be strangers. Thus, this case also ends in contradiction. Since both our claims result in contradiction, our initial assumption must be wrong, and we must have at least a group of three friends or three strangers.

This problem is, indeed, the fundamental problem of Ramsey Theory. To see why it relates to Ramsey Theory, look at the big picture. All we are trying to do is create order out of a seemingly random situation.

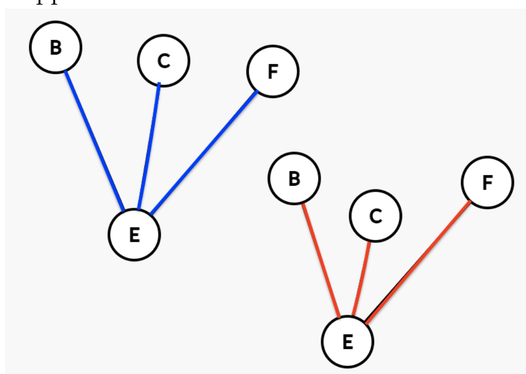
Note how we have used a graph of sorts to help us visualize this problem. Naturally, we ask the question "Does this problem have a graph-theoretic interpretation?". The answer to this question, as we will see in the next section, is yes.

3 A Graph Theoretic Interpretation

We convert this problem as follows: Say we have K_n, the complete graph on n vertices. Each of the six vertices represent a person who is attending the party. Color an edge blue if the two vertices (people) connecting it are friends, and color it red if they are strangers. What is the minimum value of k so that there will always be a blue triangle (group of three friends) or a red triangle (group of three strangers)?

Clearly this problem is the same as the one we just solved. But how do

Figure 5: Figure 5.A (top left) depicts the starting position, without loss of generality, for the case where E has 3 or more friends. Figure 5.B (bottom right) depicts the opposite — the case where E has less then three friends.



we prove n=6 is the minimum value in this case? We can do so by using the **pigeonhole principle**. Although it is assumed that the reader knows what the pigeonhole principle is, we will define it here for the sake of posterity.

Theorem 3.1. *(Pigeonhole Principle) If we are told to place* m > n *pigeons in* n *holes, then one hole must have at least 2 pigeons.*

Although the pigeonhole principle seems very simple, we will see its power in our proof as follows: Label the six vertices A, B, C, D, E, and F. Choose one vertex randomly — we will choose vertex B. Clearly, there are five edges extending out from B. By the pigeonhole principle, three of these edges must be of the same color (red or blue), since if less than three are red, then at least three are blue. Now look at the vertices at the other end of these monochromatic edges. If all of these edges are blue, we have a blue triangle, and we are done. Similarly, if one of these edges is not blue, the three edges must be all red, and we are done.

Note that we have used similar but different arguments for both of the different interpretations for this problem, and they are both quite simple. After seeing both of these solutions, it is natural to ask whether this relationship can be generalized. Indeed, the generalization of this problem is known as **Ramsey's Theorem** and was proved by Frank Ramsey in 1930.

4 Ramsey's Theorem

In this section we prove Ramsey's Theorem, and proceed to discuss its relationship with the Friends and Strangers problem.

Theorem 4.1. *(Ramsey's Theorem) Given any positive integers p and q, there exists a smallest integer n=R(p, q) such that every 2-coloring of the edges of K_n contains either a complete subgraph on p vertices, all of whose edges are in color 1, or a complete subgraph on q vertices, all of whose edges are in color 2.*

Proof. We induct on $p+q$. Our base case is $p+q$=2, which only arises if p and q are both 1. It is clear that R(1,1)=1. Assume the theorem holds whenever $p+q < N$ for some positive integer N. Let P and Q be integers such that $P+Q=N$. We have $P+Q$-1 < N, so we know that (from our inductive hypothesis), that $R(P\text{-}1,Q)$ and $R(R,Q\text{-}1)$ exist. Now look at any coloring of K_v in two colors c_1 and c_2, where v $\geq$ R(P - 1, Q) + R(P, Q - 1). Let x be a vertex of K_v. By the pigeonhole principle and because $v \geq$ R(P 1, Q)+R(P, $Q1$), we know that of the v - 1 edges that x is incident to, either R(P-1, Q) edges are in color c_1 or R(P, Q-1) edges are in color c_2. If x is incident to R(P-1, Q) edges of color c_1, consider the $K_{R(P\text{-}1,Q)}$ whose vertices are the vertices joined to x by edges of color c_1, that is the subgraph induced by the neighborhood of x. Because we know that R(P-1, Q) exists, there are two possible cases to consider. One is that this graph contains a $K_{P\text{-}1}$ with all edges in color c_1, in which case this $K_{P\text{-}1}$ together with x forms a monochromatic K_P in color c_1. The other possibility is that $K_{R(P\text{-}1,Q)}$ contains a $K_!$ with all edges in color c_2. In either case, we can see that R(P, Q) exists. A parallel argument holds if x is incident to R(P, Q-1) edges of color c_2, and K_v again contains one of the required monochromatic complete graphs. Thus, R(P, Q) exists, and in fact, because we chose v such that $v \geq$ R(P-1, Q)+R(P, Q-1), we know that R(P, Q) $\geq$ R(P, Q-1) + R(P-1, Q). [1] ■

Note that this theorem can be generalized further for any finite number of colors. Also note that Ramsey's Theorem proves *existence*, but does not give us an easy way to find R(p,q) in general.

Our Friends and Strangers problem immediately tells us that R(3,3)=6. Indeed the problem, in terms of this theorem, guarantees that there is some smallest number of people at the party required to ensure that there is either a set of p mutual acquaintances or q mutual strangers.

5 Conclusion

In this expository paper, we have presented the reader with the fundamentals of Ramsey Theory, a vast field that still holds much to explore. We hope the simplicity of the arguments presented as well as the various ideas presented inspire the reader to explore more by themselves and delve deeper intro this intriguing subject. Finally, we would like to thank Simon Rubinstein-Salzedo of Euler Circle for providing guidance while writing this paper.

References

[1] Kristina Buschur. *Introduction To Ramsey Theory*, 2011.

[2] Randall Heyman. *Intro Ramsey Theory*, 2018.