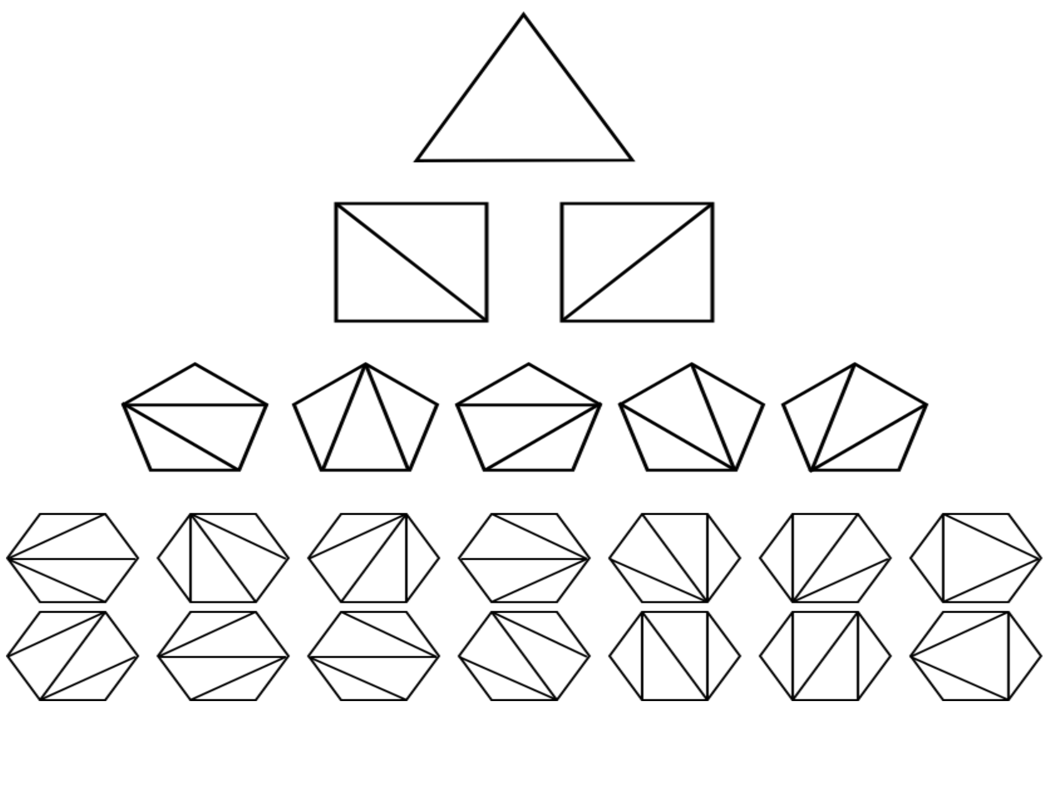
Catalan Objects

This paper is going to be on the topic of Catalan Numbers. More specifically the things counted by Catalan Numbers. Now what is a Catalan number. C_n = (1/n+1)(2nCn). Now there is a recursive function for the Catalan Numbers which is: $C_n = ((4n-2)/(n+1))C_{n-1}$ where $C_0 = 1$. But how do we get this Recursion statement? $C_n = ((4n-2)(4n-6)...(6)(2)C_0)/((n+1)(n)...(3)(2))$. We can factor out 2^n and we are left with $2^n((2n-1)(2n-3)...(1))/(n+1)!$. This simplifies to $2^n(2n!)/(2^n(n+1)!n!)$ which becomes (1/n+1)(2nCn) the outcome we desire. Now some examples of Catalan Objects.

Example 1.



Triangulation of polygons. If you have an n+2 gon for n>1, the number of ways to triangulate the polygon by taking one vertice and connecting it to all the other vertices is counted by the catalan numbers. Here is a bijective proof.

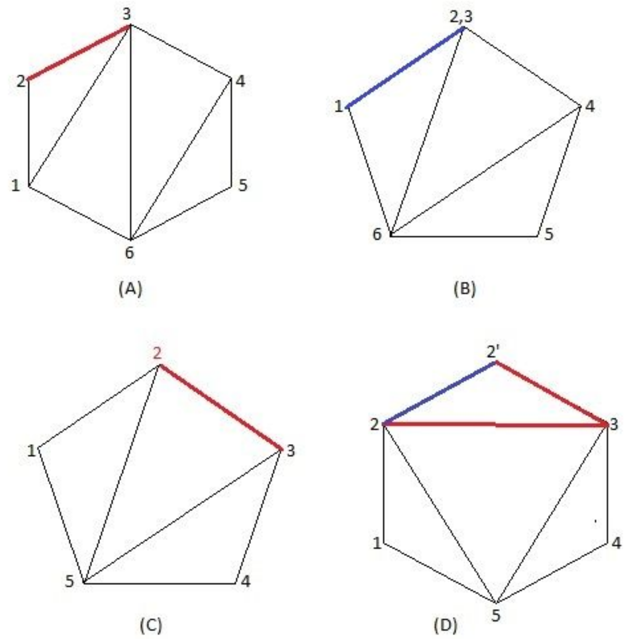
It takes n−1 diagonals to triangulate a polygon with n+2 sides. If look at is as a graph, that graph has (n+2)+(n−1)=2n+1 edges, so the sum of the degrees of the vertices is 2(2n+1)=4n+2. This is because of the sum of the degrees of the vertices theorem. Thus, there are $(4n+2)C_n$ ordered triples ⟨T,v,e⟩ where T is a triangulation of an (n+2)-gon whose vertices have been labelled 1,2,…,n+2 clockwise, v is one of its vertices, and e is one of the edges at the vertice.

Now let's look at an (n+3) gon with vertices labeled 1,2,3…(n+3). There are $(n+2)C_{n+1}$ ordered pairs ⟨T,e⟩ where T is a triangulation of the (n+3)-gon, and e is an edge of the (n+3)-gon not including the edge {1, n+3}

Figure (A) shows a triangulation T of a hexagon and the edge {2,3}, indicated in red. To get the corresponding triangulated pentagon, vertex, and edge, delete the red edge and identify its endpoints; the result is Figure (B). Here the collapsed vertex labelled 2,3 is the selected vertex, and the blue edge that resulted from collapsing edges {1,2} and {1,3} of Figure (A) is the selected edge.

The other direction of the bijection is shown in Figures (C) and (D). Figure (C) shows a triangulated pentagon with a selected vertex (2) and edge ({2,3}), each shown in red. Now split the selected vertex into two vertices, 2 and 2′, connected by a new edge, and split the selected edge into two edges, {2,3} and {2′,3}; the result is shown in Figure (D), with the new edge in blue and the split edges in red.

We can clearly see the bijection.



Another examples, but with no proof.

Example 2:

If there are 2n people sitting around a table, then there are C_n ways for the people to shake hands such that no arms cross over each other.