

# Sturmian words

By me

# What are Sturmian words?

Sturmian words are infinite sequences of bits such that there are  $n+1$  substrings of length  $n$ . For example, the Fibonacci word  $0100101001001\dots$  is a Sturmian word. The substrings of length 5 are  $01001$ ,  $10010$ ,  $00101$ ,  $01010$ ,  $10100$ , and  $00100$ . All of these substrings can be extended, but only one can be followed by both 0 and 1, and that one is  $10100$ .

# Alternative definitions

How many of the length 5 substrings of the Fibonacci word are palindromes? The answer is 2, and it's the same for every Sturmian word. In fact, Sturmianity is equivalent to having 2 palindromic substrings for every odd length and 1 palindromic substring for every even length.

# The lines, the circles, and the lines of circles

There are several ways to generate Sturmian words. One is to list the intersections of a line with irrational slope with a grid. Intersections with vertical lines are 1s and intersections with horizontal lines are 0s. Another is to take the multiples of an irrational number mod 1 and check if the value is less than a certain other value. However, this method does not always create a Sturmian word. A third way is to generate an infinite sequence of strings  $s_n$ .  $s_0$  is 1 and  $s_1$  is 0. Further members of the sequence are generated by  $s_{n+1} = s_n^{d_n} s_{(n-1)}$  where the  $d_n$  are nonnegative integers and string multiplication is concatenation. (You can swap  $s_0$  and  $s_1$  and it will still work.) These methods are all connected, but determining the connections is left as an exercise to the reader.

