COMBINATORICS WEEK 10: HYPERPLANE ARRANGEMENTS

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1. Hyperplane Arrangements

Definition 1.1. A *Hyperplane Arrangement* is an arrangement in $\mathbb{R}^d$ of n $d-1$ dimensional subspaces.

Definition 1.2. A *vertex* is an intersection of d hyperplanes.

Definition 1.3. An *edge* is an intersection of $d-1$ hyperplanes.

Definition 1.4. A j-face is an intersection of $d-j$ hyperplanes.

Definition 1.5. A *cell* is d-dimensional connected subspace which intersects 0 hyperplanes. A cell has to be the largest it can get.

Theorem 1.6. *The number of cells in a hyperplane arrangement in $\mathbb{R}^d$ assuming general position, $\Phi_d(n) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{d}$*

Proof. We can find a recursion with these numbers, then use induction. If we consider one of the hyperplanes h_n, when we remove h_n we get $\Phi_d(n-1)$. When we add h_n back in, $\Phi_{d-1}(n-1)$ cells are created, for instance in $\mathbb{R}^3$, we can look at all the intersections of h_n and the other hyperplanes to get a 2 dimensional hyperplane arrangement with $n-1$ lines. So we get a recursion, $\Phi_d(n) = \Phi_d(n-1) + \Phi_{d-1}(n-1)$. If we plug in our formula, we get

$$\sum_{i=0}^{d} \binom{n}{i} = \sum_{k=0}^{d} \binom{n-1}{d} + \sum_{j=0}^{d} \binom{n-1}{j-1},$$

which with a little algebra becomes

$$\sum_{i=0}^{d} \binom{n}{i} - \binom{n-1}{i} - \binom{n-1}{i-1} = 0,$$

which is true because of Pascal's identity, $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$. Now that we know this, we can establish base cases $\Phi_1(n)$ and $\Phi_2(1)$. Our inductive step is simply the recursion we proved earlier. ■

We can also think of an arrangement as a poset. We define $L(A)$ to be a poset consisting of all the intersections of hyperplanes in A, even the ones which intersect nothing. For $N, K \in L(A)$, $N \leq K$ when $N \supseteq K$. We could also define it the other way, but it's common practice to do it this way.

Definition 1.7. A hyperplane arrangement is said to be *central* when the intersection of every hyperplane in the arrangement is nonempty.

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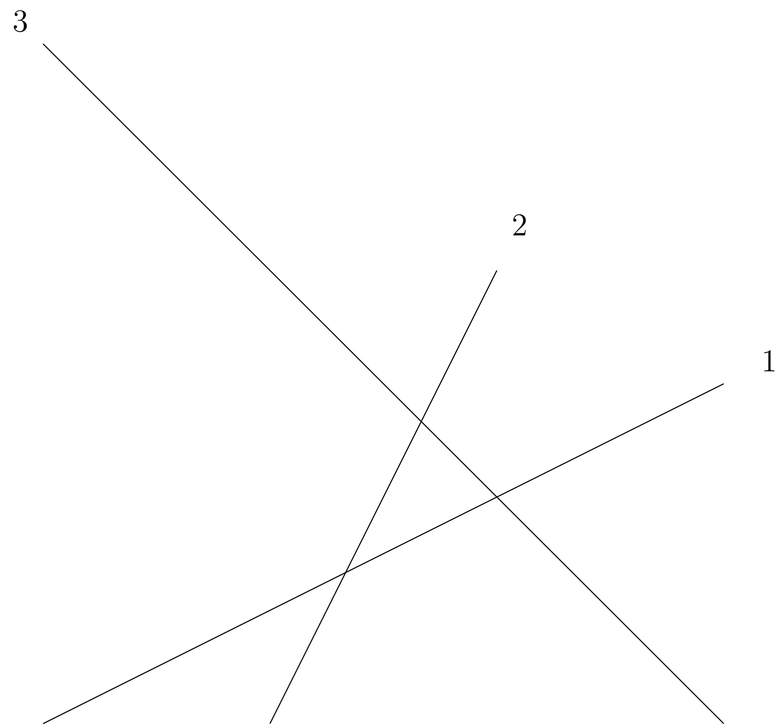

Figure 1. An arrangement of hyperplanes in $\mathbb{R}^2$

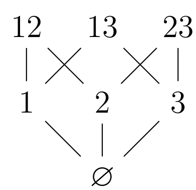

Figure 2. The Hasse diagram of $L(A)$ where A is the arrangement in Figure 1

Definition 1.8. The characteristic polynomial $\chi_a(x)$ where a is an arrangement is defined to be

$$\chi_A(x) = \sum_{\substack{B \subseteq A \\ B \text{ central}}} (-1)^{\#B} x^{d - \text{rank}(B)}$$

where A is in a d-dimensional space

Example. Let's try finding the characteristic polynomial of the arrangement in Figure 1. To do this, we can simply create a table of the value in the polynomial of each central subset, then add them all up.

B	$(-1)^{\#B}x^{d-\text{rank}(B)}$
$\varnothing$	x^2
1	$-x$
2	$-x$
3	$-x$
12	1
13	1
23	1

When we sum all of the right side up, we get $x^2 - 3x + 3$.

Theres also an interesting proposition we can make with the characteristic polynomial.

Proposition 1.9. *If A is an arrangement, then we choose a hyperplane H in it and say A' is A with H removed. We also define A'' to be the arrangement inside H whose hyperplanes are intersections of t and H where t is a hyperplane in the arrangement that intersects H. We have*

$$\chi_A(x) = \chi_{A'}(x) - \chi_{A''}(x)$$

Proof. We can start by splitting up $\chi_A(x)$ in to two parts, which are

$$\sum_{\substack{B\subseteq A\\ H\in A\\ B\text{ central}}} (-1)^{\#B}x^{d-\text{rank}(B)} + \chi_A(x) + \sum_{\substack{B\subseteq A\\ H\notin A\\ B\text{ central}}} (-1)^{\#B}x^{d-\text{rank}(B)}$$

. We can clearly see that the second one is $\chi_{A'}(x)$. Now we can look at the first one. If B is a central arrangement containing H, then B'' must be a central arrangement of A''. So we can rewrite the sum as

$$\sum_{\substack{B''\subseteq A''\\ B''\text{ central}}} (-1)^{\#B}x^{d-\text{rank}(B)} =$$

$$\sum_{\substack{B''\subseteq A''\\ B''\text{ central}}} (-1)^{\#B''+1}x^{(d-1)-\text{rank}(B'')} = -\chi_{A''}(x)$$

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