Homomesy

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1 Definition

We say (S, f, τ) is homomesic if for $f\colon \mathcal{S} \to \mathbb{R}$, $c \in \mathbb{R}$, $x = \tau^m x$, and $x \in \mathcal{S}$, for all $x \in \mathcal{S}$,

$$\frac{f(x) + f(\tau x) + ... + f(\tau^{m-1} x)}{m} = c.$$

Here, (S, f, τ), are referred to as being c-mesic.

2 Example 2.1 Inversions on a Permutation

One simple example of a homomesy are inversions on all permutations of n elements. Let us define $\mathcal{S}$ to be the set of all permutations of $[n]$ and let τ reverse the aforementioned permutation (i.e. $\tau(2, 1, 3, 5, 4) = (4, 5, 3, 1, 2)$). Let function f count the number of inversions in a permutation $\mathcal{S}_n$

Definition 2.1. *Let $\mathcal{S}_n = (s_1, s_2, \ldots s_n)$ be a permutation of n comparable, distinct elements. A pair of elements (s_i, s_j) is said to be an inversion when $s_i > s_j$ but $i < j$. For example in the permutation $(3, 1, 2, 5, 4)$, the pair of elements $(3, 2)$ would be considered an inversion.*

Observe that $x = \tau^2(x)$ meaning that all permutations $\mathcal{S}_n$ have an orbit of two. Therefore we want to show that ...

$$\frac{f(\mathcal{S}_n) + f(\tau(\mathcal{S}_n))}{2} = c$$

... for all $\mathcal{S}_n \in S$. Through observation we see that all of the inversions in S become non-inversions in $\tau(s)$ and all of the non-inversions in S become inversions in $\tau(s)$. Therefore between S and $\tau(S)$ all pairs of elements are inversions. Thus

$$c = \frac{\binom{n}{2}}{2} = \frac{(n)(n-1)}{4}$$

... and (S, f, τ) are $\frac{(n)(n-1)}{4}$ - mesic.

3 Ballot Theorems

Let a, $b \in \mathbb{N}$, $n = a+b$, $\mathcal{S}$ is the set of all words of $(s_1, s_2, \ldots, s_n)$, where a letters are equal to -1 and b letters are equal to $+1$. Each word is counting ballots in a two-way election, a for candidate A, b for candidate B. If $a < b$, B is winner once $a+b$ ballots are counted.

Q: What probability that at every stage in the counting of the ballots, candidate B is in the lead?

A: It is the same as the expected value of $f(s)$, where $f(s) = 1$ if $s_1 + s_2 + \cdots + s_i > 0$ for $1 \leq i \leq n$ where s is chosen at random from $\mathcal{S}$. (Note why this is true: f must be positive for B to be winning.)

By Bertrand's Ballot Theorem, the expected value of $f(s) = \frac{b-a}{b+a}$.

Theorem 3.1 (Bertrand's Theorem)**.** *Call a ballot sequence dominating if B is strictly ahead of A throughout the counting of votes. Any sequence of a A's and b B's, where $b > a$ has $b-a$ dominating cycle permutations. To see, arrange $a+b$ A's and B's in a circle and remove adjacent pairs AB until only $b-a$ B's remain. Each of these B's was the start of a dominating cyclic permutations before anything was removed. So $b-a$ out of the $a+b$ cyclic permutations of any arrangement of b B votes and a A are dominating.*

Now, getting to the homomesy, let $\tau\colon = C_L\colon S \to S$ be a leftward cyclic shift $(s_1, s_2, \ldots, s_n)$ to $(s_2, s_3, \ldots, s_n, s_1)$. Then

$$\frac{1}{n-1} \sum_{\mathcal{S}} f(s) = \frac{b-a}{b+a} = c$$

Therefore, (S, f, τ) are $\frac{b-a}{b+a}$ - mesic.

4 Bulgarian Solitaire

Another example of a homomesy, is the game of Bulgarian Solitaire In the game, we start off with n cards which are split into k-piles. A turn is defined to be taking one card from each pile and creating a new pile. Generally, we write the piles in order of decreasing size. An example of a turn would consist of $(5, 4, 2, 1)$ going to $(4, 4, 3, 2, 1)$. Let $\mathcal{S}$ be all partitions of our n cards, τ be one move in Bulgarian Solitaire, and f be the number of piles in an recurrent or looping cycle of positions. For us to see the homomesic property of this game, it is important that we show that all positions of Bulgarian Solitaire, eventually loop.

Theorem 4.1 (Drensky)**.** *When the total number $n = \frac{k(k+1)}{2}$ of cards is triangular, the Bulgarian solitaire will converge into piles of size $1, 2, ..., k$.*

Proof. We begin by using the *cradle model.* Take any position $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_c) \vdash n$ and create a Ferrer's board where the first column contains λ_1 boxes and so on and so forth.

We then rotate the board 45 degrees counterclockwise to create a "cradle". Assuming that λ has material points we can say that the potential energy of the system is...

$$U(\lambda) = mg\ \Sigma h_{ij}$$

...where h_{ij} is the height of a box in the i-th column to the right and j-th row up and m and g are constants. Clearly h_{ij} is proportional to $i + j$. We can now interpret a move of Bulgarian Solitaire to be taking the c boxes in the bottom row and rotating them clockwise to be the first row. When we shift the c boxes, the potential energy of the system does not change as each of the c boxes go from $(k, 1)$ to $(1, k)$ and the other $n - c$ boxes move from (i, j) to $(i + 1, j - 1)$. As a result, if $c \geq \lambda_1 - 1$, the game will cycle at the same potential energy. However, if $c < \lambda_1 - 1$, some boxes will slide down due to gravity and decrease the board's potential energy. Since the partitions of n are finite, we will at some point reach a place where moves do not decrease the system's potential energy. At this state, the first r levels of a Ferrer's board with heights $i + j = 2, 3, \ldots, r + 1$ will be filled with $1, 2, 3 \ldots, r = \frac{r(r+1)}{2}$ boxes, respectively. If $r = k$ then we have reached the stable position $(1, 2, \ldots, k)$. However if $r \neq k$ then $n - \frac{r(r+1)}{2} = \frac{k(k+1)}{2} - \frac{r(r+1)}{2} \geq k > r$ meaning that the $r + 1$-th level has an empty space and the $r + 2$-th level has at least one box. Following these empty space and box through a finite number of moves, we see that at some point the box from level $r + 2$ will in fact slide down, which is a contradiction to the statement that we were at a state where the potential energy could not be decreased. ■

Using this, we can also show that the general case will also converge to a looping cycle. Let n have the form $n = (k - 1)k/2 + r, 0 < r \leq k$. Then in the interpretation of the cradle model the solitaire will converge with a cycle of partitions which consists of the triangular partition as bottom and r surplus blocks cycling above.

Knowing this , we see that the orbit of the looping game is k and that each of the r surplus boxes contributes to one extra pile once every orbit. Since there are always $k - 1$ piles due to the filled $k - 1$ levels, we see that (S, f, τ) are in fact $((k - 1) + \frac{r}{k})$-mesic.

5 Non Crossing Partitions

This example was detailed in a paper co-authored by Simon. Some notation will have to be introduced. A partition of $[n]\colon = \{1, 2, \ldots, n\}$ is a collection π of disjoint sets $B1, B2, \ldots, BK$ with union $[n]$. The B_i's are called Blocks. A partition π is noncrossing if whenever $1 \leq i < j < k < \ell \leq n$, we do not have i and k belonging to one block of π with j and ℓ belonging to a different block. A helpful visual interpretation of this is well detailed in Simon's paper, but will be skipped in this paper.

Let $NC(n)$ denote the set of noncrossing partitions of $[n]$. Given a pair (i, j) with

$1 \leq i < j \leq n$, the toggle operation $\tau_{i,j}$ on $NC(n)$ is defined to be

$$\tau_{i,j} = \begin{cases} P \bigcup \{(i,j)\} & (i,j) \notin P \text{ and } P \bigcup \{(i,j)\} \in NC(n), \\ P \setminus \bigcup \{(i,j)\} & (i,j) \in P, \\ P & \text{otherwise.} \end{cases}$$

This can be succinctly summarized as "if block (i, j) is present, then remove it. If it is missing, add it if possible." It is clear then that each toggle is an involution. Toggle operations acting on each other are read from right-to-left such that $\tau_{i,j}\tau_{k,\ell} \colon = \tau_{i,j} \circ \tau_{k,\ell}$.

Let P be a nonempty noncrossing partition. The arc count statistic $\alpha(P)$ of a noncrossing partition is the number of pairs (i,j) with $1 \leq i \leq j \leq n$ appearing in P.

An element $w \in W_n$ is called a partial Coxeter element if it can be written as $w = \tau_{a_k}\tau_{a_{k-1}} \dots \tau_{a_1}$, where each τ_{a_i} is a toggle at some block, and $a_i \neq a_j$ if $i \neq j$. In other words, each arc appears as a toggle in w at most once, but some might not appear in w at all.

Finally, the homomesy can be defined. Let $w \in W_n$ be any partial Coxeter element that contains every toggle of the form $\tau_{i,i+1}$. Then the triple $(NC(n), \alpha, w)$ is $\frac{n-1}{2}$ -mesic.

Proof: Not included in this paper. It is rather long and detailed to great extent in Simon's paper. Read about it there!

Applications of this homomesy, however, can be detailed while largely avoiding discussion of the proof. Let n be even and $w \in W_n$ be any partial Coxeter element that contains every toggle of the form $\tau_{i,i+1}$. Then each w-orbit of $NC(n)$ contains an even number of noncrossing partitions.

Proof. The arc count of any noncrossing partition is an integer. Therefore, the only way for the average arc count across an orbit to be $\frac{n-1}{2}$, which is not an integer for even n, is if the orbit contains an even number of noncrossing partitions. ■

This is an application of homomesies for a result completely independent of homomesies. Furthermore, there is no known way to prove this result without homomesies, emphasizing their significance.