MATROIDS

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1. Independence

Matroids are based on the idea of independence. The first example of indpendence is from linear algebra.

Definition 1.1. A finite set of vectors I is defined to be *linearly dependent* if there are some vectors $v_1, v_2, \cdots, v_m \in I$, and some scalars $a_1, a_2, \cdots, a_m \neq 0$, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0$$

It is called linearly independent otherwise. Finite independent subsets satisfy these properties:

(1) Every subset of an independent set is independent.

(2) If I_1 and I_2 are independent sets, with $|I_1| < |I_2|$, then $I_1 \cup x$ is independent for some $x \in I_2 \setminus I_1$.
(3) If I is a set of vectors, then the maximal independent subsets of I are all equal in size.

Graph theory also has a notion of independence.

Definition 1.2. A finite set S of edges is *independent* if it contains no cycles, and dependent otherwise.

We then have the same exact properties.

(1) Every subset of an acyclic set of edges is acyclic.

(2) If I_1 and I_2 are sets of acyclic edges, with $|I_1| < |I_2|$, then $I_1 \cup x$ is acyclic for some $x \in I_2 \setminus I_1$.
(3) If S is a set of edges, then the maximal acyclic subsets of S are all equal in size.

This similarity between independence leads us to create the matroid.

Definition 1.3. A *matroid* is a finite set E with a non-empty collection I of subsets of E, called independent sets, such that

(1) Every subset of an independent set is independent.

(2) If I_1 and I_2 are independent sets, with $|I_1| < |I_2|$, then $I_1 \cup x$ is independent for some $x \in I_2 \setminus I_1$.

The third property of independent sets is equivalent to the second. If we assume the second property, we could just apply the second property on two maximal indepedent sets that differ in size. If we assume the third property, then take a set S to be the union of

Date: December 24, 2017.

two independent sets, I_1, I_2 such that $|I_1| < |I_2|$. Then I_1 cannot be a maximal indepedent subset of S, so some element of $I_2 \setminus I_1$ is able to be added. Therefore, a matroid could also be defined with the independent sets satisfying:

(1) Every subset of an independent set is independent.

(2) If $S \subset E$, then the maximal independent subsets of S are all equal in size.

2. Greedy Algorithms

Definition 2.1. A maximal independent set of a matroid is called a *basis*.

Matroids have an interesting connection with greedy algorithms. Let's look at Kruskal's algorithm, which finds the minimum spanning tree of a graph. Kruskal's algorithm is as follows:

(1) Intialize a set of edges $I = \emptyset$

(2) Sort the edges by weight

(3) Iterate through the edges, starting with the cheapest one and adding an edge e to I if it does not produce a cycle in I.

Theorem 2.2. *This greedy algorithm produces a minimum spanning tree.*

Proof. Clearly, the algorithm will produce a spanning tree. Let us prove that the set S it produces is optimal by proving the property that S is always contained in a minimum spanning tree. We do this using induction on the size of S. The base case when S is empty holds. Now suppose S is contained in a minimum spanning tree B, and we add some edge x to S. If $x \in B$, then S is still contained in B. Otherwise, B plus the edge x forms a cycle and there is a different edge e that is in this cycle. Then, $B - e + x$ is still a minimum spanning tree. ■

Just like acyclic graphs became the independent sets of a matroid, this algorithm can be generalized to matroids. Suppose we have a matroid M and a function w that assigns weights to each element of M. Then we want to find the basis such that the sum of the weights is minimized. The algorithm is essentially the same:

(1) Intialize a set of elements $I = \emptyset$

(2) Sort the elements of M by weight

(3) Iterate through the elements, starting with the cheapest one and addding x if $I \cup x$ is independent.

The proof of correctness can be extended similary. The interesting fact is that these algorithms characterize matroids. Suppose we have some set S with a non-empty collection F of feasible subsets and weights on each element of the set S. We should assume that any subset of a feasible subset is feasible because our greedy algorithm builds sets up one element at a time. If our greedy algorithm finds the feasible subset with minimum value, then:

Theorem 2.3. *(Rado and Edmonds) If the greedy algorithm works for any possible set of weights, the feasible sets are the independent sets of a matroid.*

Proof. We just have to prove this new set satisfies the definition of a matroid. The first one holds, so we have to prove the second, by proving the third property. By contradiction, asssume that a set S has two maximal feasible sets F_1, F_2 such that $|F_1| < |F_2|$. We can assign the weights such that we pick all the elements of F_2 first. Then we can't pick any more elements because F_2 is maximal. However, the greedy algorithm did not find the optimal basis. ∎