YOUNG TABLEAUX

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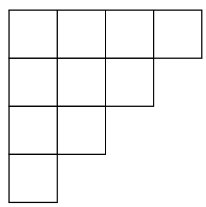
1. Young Diagrams and Young Tableaux

For an integer n, we can represent its partitions in the form of a diagram. In particular, given a partition, we can draw rows of boxes representing each element of the partition, ordered by the size of the elements. Such a diagram is known as a Young diagram.

Definition 1.1. The *Young diagram* of a partition $(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n)$ of an integer n is a set of n boxes arranged into k rows such that the length of the ith row is λ_i



11	7	1	2
14	2		
6	8		
7			

Figure 1. Young Diagram and Young Tableau

We can now place numbers within the boxes of the diagram.

Definition 1.2. A Young tableau of a given Young diagram λ is an assignment in which a positive integer is assigned to each box in λ. This tableau is of *shape* λ.

While this generic definition of a Young tableau does not yield many interesting results, as there are an infinite number of them, there are some particular subsets which are more useful and interesting.

Definition 1.3. A Young tableau is *semistandard* if all of its rows are weakly increasing and all of its columns are strictly increasing.

A *standard* Young tableau is a semistandard tableau in which the entries are the integers from 1 through n, each appearing exactly once.

1	2	4	13
7	7		
8	9		
15			

1	4	5	7
2	6		
3	8		
9			

Figure 2. Semistandard and Standard Young Tableaux

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2. Counting Young Tableaux

These standard Young tableaux are particularly interesting because we can count them. In particular, we can count the number of standard Young tableaux of shape λ. To calculate the number of tableaux, we must first define the hook length.

Definition 2.1. The *hook-length* h_b of a box b in a Young diagram is the number of boxes directly to the right or below it.

For instance, the hook length of the highlighted box in figure 2.1 is 3.

We now give the formula for the number of Young tableaux of shape λ, known as the hook-length formula.

Theorem 2.2. *(Hook length formula)*
The number f^λ of standard Young tableaux of shape λ is equal to $\frac{n!}{\Pi_b h_b}$ where Π_b represents the sum over all boxes of λ.

In other words, the number of tableaux is the number of permutations of the n boxes divided by the product of each box's hook length.

There is another useful formula, known as the ennumerative identity, which relates the different f^λ of each partition of n.

Theorem 2.3. *(Ennumerative Identity)*

$$\sum_\lambda (f^\lambda)^2 = n!$$

3. The Robinson-Schensted-Knuth correspondence

Definition 3.1. The *row-bumping algorithm* is an algorithm which, given a tableau T and a number x, yields a tableau $T \leftarrow x$ with shape one square larger than the shape of T.

The algorithm works as follows:

1. If x is greater than or equal to all of the elements in the first row, place it in a new box at the end of the first row.

2. If not, find the first element in the row that is greater than x, and replace it with x, 'bumping' the original value.

3. Repeat the process on the second row using the bumped element, continuing until either one of the bumped elements can be placed at the end of the row, or an element is bumped from the last row, in which case it is placed in the first box in a new bottom row.

The row-bumping algorithm is also invertible–given the resulting tableau and *the position of the added box*, it is possible to recover the original T and x by reversing each of the bumps.

There is one important property of two successive row-bumpings, known as the *Row-Bumping Lemma.*

In order to state the lemma, we must first define the *bumping route*:

Definition 3.2. The bumping route of a row-bumping is the collection of the boxes from which elements were bumped, combined with the newly constructed box.

With this definition, we can now state the lemma.

Lemma 3.3. *(Row-Bumping Lemma)*
Given two successive row-bumpings $T \leftarrow x \leftarrow x'$, call the routes of the two bumpings R and R', and the two added boxes B and B'. Then:

- *if $x \leq x'$, R is strictly left of R' and B is strictly left of and weakly below B'.*
- *if $x > x'$, then R is strictly right of R' and B is weakly right of and strictly above B'.*

Proof. The lemma can be proved by tracking all possible routes of the second bumping for each of the two initial conditions.

■

Given this algorithm, it is now possible to construct a Young tableau from a list of entries. This can be done by taking the tableau with no boxes and row-bumping each of the entries, in order, into the tableau. In this case, each unique 'word' of entries corresponds to one unique tableau, however, this is not a one-to-one mapping. (For example, $4 \leftarrow 1 \leftarrow 2 \leftarrow 3$ and $1 \leftarrow 2 \leftarrow 4 \leftarrow 3$ both yield the same tableau. This is because of the invertibility property of the row-bumping algorithm , which also requires the location of the added box. In order to make this process invertible, we track the order in which the boxes of the tableau were added, which comes out to be a standard tableau of the same shape as the resulting tableau. This is the *Robinson-Schensted Correspondence*:

Definition 3.4. The Robinson-Schensted correspondence is the bijective map between words w of length r with letters taken from $[n]$ and pairs of tableaux (P, Q) with:

- r boxes
- the same shape
- P semistandard with entries taken from $[n]$
- Q standard

obatained by successively row-insterting the letters of w to create P and recording the positions of each new box in Q.

Notably, if P is also standard, then the corresponding word contains each of the entries from 1 to r exactly once, making it a permutation. This leads directly to a proof of the Ennumerative Identity using the Robinson-Schensted correspondence.

The correspondence can further be generalized to one using ordered pairs of semistandard tableaux P, Q. In this case, we recover pairs of elements one at a time by finding the rightmost instance of the highest element in Q and using its position as the box on which to perform the reverse row-bumping of P. This eventually results in a two-row array $\begin{pmatrix} u_1 & u_2 & u_3 & \ldots & u_r \\ v_1 & v_2 & v_3 & \ldots & v_r \end{pmatrix}$ where the u_k are the elements of Q and the v_k are the elements of P. In addition, these arrays are lexicographically ordered; they satisfy the following two conditions:

- The u_k are weakly increasing (true by construction)
- If $u_k = u_{k+1}$, then $v_k \leq v_{k+1}$ (true by the Row-Bumping Lemma)

This construction can indeed be reversed, by succesively row-bumping each of the u_k to construct P while marking the new boxes with the v_Ks to construct Q. P is semistandard by definition. Q, on the other hand, does not appear to be, due to the possibility that two equivalent values end up vertically adjacent to each other. However, by the lexicographic property, $u_k \leq u_{k+1}$, which, by the row-bumping lemma, implies that v_{k+1} must land to the right of v_k, which is consistent with the original construction of the array.

Definition 3.5. The *Robinson-Schensted-Knuth* correspondence is the bijective map between lexicographically ordered two-row arrays $\begin{pmatrix} u_1 & u_2 & u_3 & \ldots & u_r \\ v_1 & v_2 & v_3 & \ldots & v_r \end{pmatrix}$, with u_k taken from the alphabet $[m]$ and v_k taken from the alphabet $[n]$ and pairs of semistandard tableaux (P, Q) with r boxes and the same shape, where the entries of P are from $[n]$, and the entries of Q are from m.

The map is obatained by successively row-inserting each v_k to construct P and recording the positions of each new box with u_k in Q.

Under this correspondence, the elements of Q are also the elements of the first row of the array, and the elements of P are also the elements of the second row.

In addition, if Q is standard, then the first row just becomes the integers from 1 to r, making the second row simply form a word (This is equivalent to the Robinson-Schensted correspondence).

If both P and Q are standard, then the first row is the integers from 1 to r in order, while the second row is the same set of integers in a different order, making the array a permutation.

4. Young Tableaux and Representations of the Symmetric Group

The Young Tableaux have a strong relationship with the irreducible representations of the Symmetric group S_n. In order to explain this relationship, we first must define a representation of a group.

Definition 4.1. A representation of a group G is a homomorphism $G \to GL(\mathbb{C}, n)$ where $GL(\mathbb{C}, n)$ is the group of linear transformations with determinant 1 under the operation of matrix multiplication.

Each group has a large number of representations, however, most representations can be broken down into combinations of simpler representations. We characterize these representatins as follows:

Definition 4.2. If the space acted on by the image of a representation φ contains a subspace which is invariant under the action of the image of φ, the representation φ is *reducible*. Any representation which is not reducible is *irreducible*.

One of the most important theorems of representation theory is Maschke's Theorem.

Theorem 4.3. *(Maschke's Theorem)*
All reducible representations V of a group G can be represented as a combination of irreducible representations W_1, W_2,...,W_k of G,

$$V = W_1 \oplus W_2 \oplus ... \oplus W_k$$

.

Maschke's Theorem implies that the study of representations is based upon the study of irreducible representations. However, categorizing the irreducible representations of a group is not an easy task. This is where Young tableaux come in useful, as they can be connected by the following theorem, another important result in representation theory.

Theorem 4.4. *The number of distinct irreducible representations of a group is equal to the number of conjugacy classes.*

This in particular seems to suggest a direct connection with the Young tableaux, as the conjugacy classes of the symmetric group are equivalent to all possible cycle structures of the group elements. A cycle structure splits all the permuted elements into cycles of varying sizes, so the number of possible cycle structures is the same as the number of partitions of n.

Proposition 4.5. *The number of distinct irreducible representations of S_n is equal to the number of distinct Young diagrams with n nodes (the number of partitions of n).*

Now that this relationship has been established, the next logical step is to assign every irreducible representation to a Young diagram. This can indeed be done, through what is known as a *Specht module*, which establishes a bijection between the irreducible representations and the Young diagrams.

This correspondence has an additional property:

Proposition 4.6. *The number of standard Young tableaux of shape λ is equal to the dimension of the representation which corresponds to λ.*

With this, we can now prove the hook-length formula, starting with the following lemma, which relates the hook lengths of nodes in a Young diagram.

Lemma 4.7. *Given a node (i, j), each of the integers from 1 to h_{ij} can be rewritten either as h_{it}, $j \leq t$, or as $h_{ij} - h_{sj}$, $i < s$, but not both.*

To prove this lemma, we need to define a few terms.

Definition 4.8. A hook is the set of nodes in a Young diagram directly to the right of or below a node, including the node itself. The subset of the hook in the same row as the defining node is the *arm*, and the rightmost node in the arm is the *head*. The subset of the hook in the same column as the defining node is the *leg*, and the bottommost node in the leg is the *foot*.

Definition 4.9. A rim node is a node (i, j) in a Young diagram such that there is no node $(i + 1, j + 1)$. Note that the head and the foot of a hook must be rim nodes.

Definition 4.10. Given a hook defined at (i, j), the set of all rim nodes between the head and foot of the hook, inclusive, is the *rim hook* defined at (i, j). Note that because each node of the rim hook is directly to the left or below the previous node, and the rim hook starts and ends at the same nodes as the standard hook, the number of nodes in the rim hook is also equal to h_{ij}.

It is possible to split a rim hook into two smaller pieces with a cut between two adjacent nodes. If they are vertically adjacent, then the upper-right piece does not end in a foot and is thus not a rim hook, and the lower left piece ends in a head and thus is a rim hook. If they are horizontally adjacent, the opposite holds.

If the upper right piece is a hook, then its length can be written as h_{it}, $j \leq t$, and if it is not a hook, then its length can be written as $h_{ij} - h_{sj}$, $i \leq s$. $i = s$ only when the length is zero, so the integers from 1 to h_{ij} can be rewritten either as exactly one of h_{it}, $j \leq t$ or $h_{ij} - h_{sj}$, $i < s$, proving Lemma 4.7.

We can now prove the hook-length formula.

Proof. By this lemma, the products of all the hook lengths in the ith row of a Young tableau is equal to the product of all integers from 1 to h_{i1} which cannot be written as $h_{i1} - h_{s1}$, $i < s$, or

$$P(i) = \frac{h_{i1}!}{\prod_{s>i}(h_{i1} - h_{s1})}$$

.

From this, we can prove the hook-length formula using a result from representation theory which states that states that the degree of a representation corresponding to a partition λ is equal to

$$n!\prod_i \frac{1}{l_i!}\prod_{s>i}(l_i - l_s)$$

where l_i is equal to $\lambda_i + k - i$. This value is equal to h_{i1}, so the formula can be simplified to

$$n!\prod_i \frac{1}{h_{i1}!}\prod_{s>i}(h_{i1} - h_{s1})$$

.

$$= n!\prod_i \frac{1}{P(i)}$$

$$= \frac{n!}{\prod_b h_b}$$

which is the hook-length formula. ■

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