Palindromes in Sturmian Words

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1 Combinatorial Words

Definition 1. An alphabet A is a nonempty set of finite symbols like $A : \{a, b\}$.

Definition 2. A word W is a sequence of symbols from A

Definition 3. The product $a \cdot b$ of the words $a = a_1a_2 \ldots a_m$, $b = b_1b_2 \ldots b_n$ is the word $a_1a_2 \ldots a_mb_1b_2 \ldots b_n$.

Definition 4. A word u is a factor of w (resp. left factor or a prefix, a right factor or a suffix) if there exist words x and y such that $w = xuy$ (resp. $w = uy$, $w = xu$).

2 Palindromes in Sturmian Words

Definition 5. An infinite word x is a Sturmian word if and only if there are $n + 1$ distinct factors of length n for all positive intereger n.

It follows from the definition above that Sturmian words are composed entirely of two symbols. We will use a, b to denote these symbols.

Theorem 6. *For any positive integer n, there is a unique factor w of the a Sturmian word x such that both wa and wb are both factors of x.*

Theorem 7. *An infinite binary word x is Sturmian if and only if the following conditions are met.*

- The difference between the number of a's in any two factors of equal length (of the x) is at most 1.
- The word x is not ultimately periodic.

Theorem 8. *If x is a Sturmian word and w is a factor of x, then w occurs infinitely many times within x.*

Theorem 9. *If x is a Sturmian word and w is a factor of x, then $\bar{w}$, the mirror image of w, is also a factor of x.*

Theorem 10. *If x is a Sturmian word and n is a positive integer, the number of palindromic factors of length n (of the word x) is:*

- 1 if n is even
- 2 otherwise

An inductive proof of Theorem 10 can be found in [1].

References

[1] Xavier Droubay and Giuseppe Pirillo, *Palindromes and Sturmian words*, Theoretical Computer Science 223 (1999) 73-85.