Interesting Polytopes

Aditya Mittal

California, United States

Abstract

In math, most shapes can be considered a polytope, or a shape with finite faces, each with zero curvature. The three that I will be discussing today, will be two constructed using and with combinatorial, and topological properties: The Associahedron, and the Permutohedron,.

1. The Associahedron

The associahedron, for every integer $n \geq 2$, K_n, is defined to be an $(n-2)$-dimensional convex polytope, having each vertex correspond to a way of correctly parenthesizing a word of n letters, and each vertex corresponds to a single use of the associativity rule (that being where the name is from). This can then also be seen as that each vertex is a triangulation of a $n+1$ sided regular polygon, with edges representing edge flips, where a diagonal is removed from the triangulation, and replaced by a different one. So if two triangulations of the polygon, differ by one diagonal, then they will be adjacent in the associahedron. We can then conclude that the amount of vertices contained in each higher dimension associahedron, will be the C_{n-1} Catalan number.

A nice way to think about these shapes is through the idea of *loop concatenation* (this is why associadra are also called loop spaces). This would be defined as taking a loop, and imagining the length of it through an interval of 0 to 1 . If you were to have two loops (it would resemble an infinity sign; the representation of the K_2), loop a and loop b, and concatenated them, you would have a continuous shape that started and ended at the same point. We'll then say that the length of this object still spans our interval of 0 to 1. Regardless of how we split it, it still represents the same thing of $a * b$. Only once we reach K_3 do things start to get interesting. K_3 represents three loops, a, b, and c. Concatenating them, we get a three leaf clover. If we were to think about the ways of parenthesizing this, we would get $(ab)c$ and $a(bc)$. The first one represents having to go around both loops a and b in the first half, and c in the remaining time. The second representing a similar idea: a in the first half second, and b and c in the remaining time. Although these seem like very different things (and to be honest, they are), there is a way of connecting the two: Take a continuous function that simultaneously slows down the time for c while accelerating a. And since this is continuous, there are an infinite amount of points/other 3 leaf clovers between $(ab)c$ and $a(bc)$. So, we represent this with a line segment, with end points being these two parenthesizations. That's our use of the associativity rule as stated before! These changes are known as a *homotopy*, denoted by $H(a, b)$, with a and b are two vertices of our "clovers".

This is where the connection between topology and combinatorics shows up. You can think of the C_{n-1} Catalan number of the associahedron, K_n as our topology, and the quantity as our subsets comprising it. For example, with the K_3 associahedron, we have the $C_2 = 2$ vertices (our topology) constructing our associahedron. Each vertex (or parenthesization) represents a subset constructing our topology. The previously stated homotopies is what then makes our topology continuous, creating the set of neighborhoods for each vertex. It shows the infinite amount of superpositions a vertex can move between, to move to another vertex.

So far we have looked at a $0-dimensional$ and a $1-dimensional$, with K_4, we get a 5 vertices (and thus forming a pentagon). If we were to look at the pentagon's vertices, we would have 2 homotopies. Now what's really cool about this, I would be able to find an infinite amount of connections from these homotopies, to the other homotopy. If I represent these two homotopies as parallel lines, I could pick any point along one line, and connect it to any point along the other. This means that the pentagon $itself$ is a homotopy. It's the associativity rule being applied again to be projected onto something else that has already been affected by the associativity rule.

This also translates into higher dimensions: For K_5 (the 3-dimensional associahedron), the vertices are paranthesizations, edges are homotopies between vertices, faces are homotopies of homotopies (the edges), and the solid interior is all of the homotopies of the homotopies (the faces) of the original homotopies (the edges).

This is a basic idea of what associahedra are supposed to represent.

2. The Permutohedron

The *permutohedron* (sometimes spelled, *permutahedron*) of order n, is an $n-1$ dimensional polytope, whose vertices are given by permuting the coordinates $[n]$ (i.e. the permutohedron of order 4, is a 3-dimensional object with 4! (24) vertices, and it's embedded in 4-dimensional space), while also being embedded in n-dimensional space. This being embedded is important to note due to its topological relevance (this also means that the permutohedron of order n lies in the $(n-1)$-dimensional hyperplane).

An embedding in topology is defined to be a function $f : X \to Y$ where X and Y are topological spaces, and f forms a homeomorphism (a continuous function) between X and $f(X)$ ($f(X)$ also then gains the subspace topology from Y, meaning that the $f(X)$ is a subset of Y, and its topology being derived from Y). With that in mind we can say that if we embed a permutohedron of order n into an n-dimensional space, then that if it was an object of that dimension (instead of being embedded, just "existed"), it would retain all of its topological properties, as this mapping can also be reversed with an inverse function, $f^{-1}(X)$.

Another thing about these shapes is that each vertex, is only connected to $n-1$ other vertices. This then tells us that there are $\frac{n!(n-1)}{2}$ edges, while being length $\sqrt{2}$. These edges connect vertices whose permutation only differs by swapping two coordinates in which the values differ by 1. So the coordinate of 1, can only be swapped with 2 to be connected with an edge. 2 can be swapped with 1 or 3. 3 with 2 or 4. And 4 with only 3. (This is with the permutohedron of order 4) There is also one face per nonempty subsets, S, of $[n]$, consisting of the vertices in which all coordinates in positions in S are smaller than all coordinates in positions not in S. That means, the total number of faces is $2^n - 2$, as you can either choose to keep or not to keep each number (that's where the 2 comes from), and you do this n times. The subtraction of 2 comes removing the empty set, and the set itself (as those aren't proper subsets). Also, one can generalize this, as there is a bijection on the *strict weak orderings* of the set of $[n]$, in which there are $n - d$ equivalence classes. Because of this, the number of faces is also given by the *ordered Bell numbers*.

Another thing to note about permutohedra is that they are *isogonal*, or vertex-transitive. This is because each coordinate of a vertex in a given permutohedron is **acted** upon by the *symmetric group*.

The *symmetric group*, S_n, is the group of all permutations of $[n]$ with order $n!$, and every subgroup of S_n is of order n.

The set being **acted** upon is important to note: A group action is when a group acts on a set, and permutes its elements, such that the mapping from the group to the *permutation group* of the set forms a *homomorphism*. Since we're acting upon our set with the symmetric group, the permutation group is just a subgroup of the symmetric group, meaning that there intrinsically is a homomorphism between the two, preserving the structure of our vertices (which is our set that we're acting upon). This is why permutohedra are isogonal.