Mancala Game Combinatorics

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1 Introduction

Mancala is a family of games, which are typically two-player. Mancala games have boards made from some number of pits, which hold objects such as stones, seeds, or beans. The type of object placed in the pits is not important, so we will call them all stones. The goal of a mancala game is usually to collect more stones than your opponent. Unlike most games studied in combinatorial game theory, mancala games have a score for each player.

2 Ayo

Definition 1. *Ayo is a mancala game originating from the Yoruba people. Ayo is played on a $2 \times n$ board where each player gets one side. The typical starting position has four stones in each pit. The rules are as follows:*

1. *When either player cannot move on their turn, the game will end and that player will get all of the remaining stones. Additionally, if there are too few stones for any captures to be made and it is impossible to end the game through running out of legal moves, the game will end and both players will get the stones on their side of the board.*

2. *Each player can* ***harvest*** *(move the stones in) a pit on their side of the board. When a player harvests a pit with less than $2n$ stones, they will sow (place 1 stone in) each pit going counter-clockwise until they run out of stones. If the player harvests a pit with at least $2n$ stones, which is called an* ***odu****, they will do the same thing, but skip the first pit on either side.*

3. *Any move a player makes must give their opponent a move that satisfies all of the other conditions.*

4. *If the last pit sown from a player's move is on their opponent's side of the board and it contains either 2 or 3 stones, they will capture the stones in that pit. When this happens, the player will additionally capture the stones of any other pits sown that are on the opponent's side and contain 2 or 3 stones.*

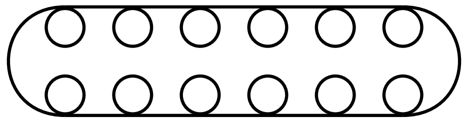

Figure 1: A typical Ayo board

The pits on left's side of the board are numbered clockwise from 2 to $n-1$, and the pits on right's side of the board are number counter-clockwise from 1 to $2-n$. Although this numbering seems irrational, it will make it easier to equate Ayo positions with Tchoukaillon positions later on.

Definition 2. *A determined position is an Ayo position that satisfies the following conditions:*

1. *Left will make a capture on every turn until the end of the game.*
2. *No move can be made from an odu*
3. *There will always be exactly one stone on right's side of the board*
4. *Left will capture all but one stone*

Theorem 1. *The stone on right's side of the board in a determined position must be in pit* 1 *on right's turn and pit* 0 *on left's turn.*

Proof. When left moves, she must make a capture while leaving only 1 stone on right's side of the board. For this to happen, she must sow every pit with a stone on right's side of the board as well as exactly 1 empty pit. Because right's side of the board has 1 pit with a stone in it, she will sow 2 pits, which must be pits 0 and 1. If the stone is in pit 1, then it will be in pit 0 on right's turn. Right will then move the stone to -1, which means the game is no longer a determined position, which means the original position was not a determined position. Therefore, the stone must be in pit 0 and will be in pit 1 on right's turn.

Due to the reasoning above, the stone on right's side must end up in either pit 0 or pit 1. For this to occur, right must move the stone from 1 to 0, which means the stone must be in pit 1 on right's turn and will be in pit 0 on left's turn. □

3 Tchoukaillon

Definition 3. *Tchoukaillon is a solitaire variant of the Russian mancala game Tchouka. Both games are played with a board containing a pit called the* ***Roumba*** *and n pits going out from the Roumba numbered* 1 *through n. The goal of Tchoukaillon is to put every stone in the Roumba. The rules of Tchouka are as follows:*

1. *When it is a player's turn to move, they will harvest a pit and sow the stones in each pit towards the Roumba until they run out of stones. If the Roumba is reached, the sowing direction will be reversed. If the pit opposite the Roumba is reached, the direction will be reversed again.*

2. *If the last pit sown is the Roumba, the player will make another move.*

3. *If the last pit sown is a non-empty pit besides the Roumba, the player will immediately harvest that pit.*

4. *If the last pit sown is empty, the player's turn is over.*

Tchoukaillon has the same movement, except that the direction of sowing cannot be reversed and the last stone must land in the Roumba.

There is a simple way to equate Tchoukaillon positions with determined Ayo positions. Tchoukaillon is easier to analyze than Ayo, so establishing this connection is helpful for analyzing Ayo.

Theorem 2. *The only winning move in Tchoukaillon, if it exists, is to harvest the smallest possible pit.*

Proof. For pit n to be harvestable, it must have exactly n stones. If it has less than n stones, the last stone will not fall in the Roumba. If it has more, then the stones would have to go past the Roumba, which is not allowed. Because the number of stones in a pit cannot decrease, any winning move must not result in some pit n having more than n stones. Therefore, if pits i and j are both harvestable where $i > j$, moving in pit i is not a winning move, as j will have $j + 1$ stones. □

Theorem 3. *The sets of determined positions in Ayo and winnable position in Tchoukaillon have a one-to-one correspondence, and the corresponding Tchoukaillon position for any Ayo position is the position where pit n has the same number of stones as the nth pit in the Ayo position for all $n > 0$.*

Proof. Let A be an Ayo position and T be the corresponding Tchoukaillon position. First, we will show that each move in A has a corresponding move in T By theorem 1, this can be split up into two cases:

1. When it is right's turn to move, there will be a stone in pit 1. Right must move this stone to pit 0, which has the corresponding move in T of moving a stone from pit 1 to pit 0.

2. When it is left's turn to move, she must capture the stone in pit 0. For this to happen, she must harvest some pit n that has n stones. Additionally, this must be the smallest possible n to prevent moving to a position that is not determined. The corresponding move in T is to move the stones in pit n. In both, one stone is sowed in each pit from 1 through $n - 1$.

This means that the only move that will be made is in the smallest n such that pit n contains n stones, and that the corresponding move in T is to move the stones in pit n.

The strategy is the same in T. The player must harvest some pit n that contains n stones to sow the last stone in the Roumba. They also must harvest the pit with the smallest such n. The corresponding move in A is moving the stones in pit n.

For Tchoukaillon and Ayo positions to be winnable or determined respectively, they must be or be able to reach a position where there is a stone in pit 1 and no stone in any pit $n > 1$. Because the condition is the same, the corresponding Tchoukaillon position for any Ayo position must be winnable. Similarly, any winnable Tchoukaillon position will have a corresponding Ayo position. □

4 Winning Positions

Theorem 4. *There is exactly one winning position with n stones for all $n \geq 0$.*

Proof. A position must have a move to a winning position to be a winning position. This means that we can work backwards to prove this. Let $p_n(i)$ be the number of stones in pit i in the position with n stones where $i > 0$. $p_0(i)$ is 0 for all i.

For any n, the pit that was previously moved in to reach the position with n stones must be a pit i such that $p_n(i) = 0$, since moving the stones in pit i will remove all of the stones from the pit. It also must be the pit with the smallest such i. Otherwise, any pit $j < i$ such that $p_n(j) = 0$ would have at least one stone. Because the stones every pit with number $j < i$ will have a stone added when the stones in i are moved, one stone must be subtracted from each to get the position with $n + 1$ stones. Therefore, the position with $n + 1$ stones is

$$p_{n+1}(i) = \begin{cases} p_n(i) - 1 & i < \min\{j : p_n(j) = 0\} \\ i & i = \min\{j : p_n(j) = 0\} \\ p_n(i) & i > \min\{j : p_n(j) = 0\}. \end{cases}$$

□

Let $m_n(i)$ be the total number of times pit i is harvested before winning in the position with n stones and let $b_n(i)$ be the total number of times pit i is sown before winning in the position with n stones. Note that $p_n(i) = im_n(i) - b_n(i)$. Also note that $b_n(j) = \sum_{j>n}$.

Theorem 5. $(p_n(1), p_n(2), \ldots, p_n(i)\}$ *repeats as n increases with a period of* $lcm(1, 2, \ldots, i + 1)$.

Stones	Pit 1	Pit 2	Pit 3	Pit 4	Pit 5	Pit 6	Pit 7	Pit 8	Pit 9
1	1								
2	0	2							
3	1	2							
4	0	1	3						
5	1	1	3						
6	0	0	2	4					
7	1	0	2	4					
8	0	2	2	4					
9	1	2	2	4					
10	0	1	1	3	5				
11	1	1	1	3	5				
12	0	0	0	2	4	6			
13	1	0	0	2	4	6			
14	0	2	0	2	4	6			
15	1	2	0	2	4	6			
16	0	1	3	2	4	6			
17	1	1	3	2	4	6			
18	0	0	2	1	3	5	7		
19	1	0	2	1	3	5	7		
20	0	2	2	1	3	5	7		
21	1	2	2	1	3	5	7		
22	0	1	1	0	2	4	6	8	
23	1	1	1	0	2	4	6	8	
24	0	0	0	4	2	4	6	8	
25	1	0	0	4	2	4	6	8	
26	0	2	0	4	2	4	6	8	
27	1	2	0	4	2	4	6	8	
28	0	1	3	4	2	4	6	8	
29	1	1	3	4	2	4	6	8	
30	0	0	2	3	1	3	5	7	9
31	1	0	2	3	1	3	5	7	9
32	0	2	2	3	1	3	5	7	9

Figure 2: This is the table of stones in each pit based on the total number of stones up to 32. Empty cells mean 0. The values are known up to $21,286,434$ stones.

Proof. To prove this, we will show by induction that $t = lcm(1, 2, \ldots, i+1)$ is the smallest positive number such that $p_t(j) = 0$ for all $1 \leq j \leq i$. Because the contents of pits 1 through i are not affected by the contents of the pits numbered $i+1$ or higher, proving this will also prove the theorem. This is true for $i = 1$, since $lcm(1, 2) = 2$, $p_1(1) = 1$, and $p_2(1) = 0$.

Assume that this is true for all $1 \leq j < i$ for some $i > 1$. Let $t = lcm(1, 2, \ldots, i+1)$. Also let $1 \leq j < i$ and $k \in \mathbb{N}^+$. Because kt is a multiple of $lcm(1, 2, \ldots, j+1)$, $p_{kt}(j) = 0$. This means that

$$\begin{aligned} jm_{kt}(j) &= p_{kt}(j) + b_{kt}(j) = b_{kt}(j) = b_{kt}(j+1) + m_{kt}(j+1) \\ &= (j+1)m_{kt}(j+1) + m_{kt}(j+1) = (j+2)m_{kt}(j+1) \end{aligned}$$

Applying this repeatedly gives that $2m_{kt}(1) = i(i+1)m_{kt}(i)$. Because every other move is a move in pit 1 and kt is even, $2m_{kt}(1) = kt$. This means that

$$\begin{aligned} p_{kt}(i+1) &= (i+1)m_{kt}(i+1) - b_{kt}(i+1) \\ &\equiv (i+1)(m_{kt}(i+1) + b_{kt}(i+1)) \pmod{i+2} \\ &\equiv (i+1)b_{kt}(i) \pmod{i+2} \\ &\equiv i(i+1)m_{kt}(i) \pmod{i+2} \\ &\equiv 2m_{kt}(1) \pmod{i+2} \\ &\equiv kt \pmod{i+2} \end{aligned}$$

Because $0 \leq p_{kt}(i+1) \leq i+1$, $p_{kt}(i+1) = 0$ iff $kt \equiv 0 \pmod{i+2}$. The smallest value of k that will make this true is $(i+2)/gcd(t, i+2)$. With this value of k,

$$kt = \frac{(i+2)t}{gcd(t, i+2)} = lcm(t, i+2) = lcm(1, 2, \ldots, i+2)$$

This means that for all i, $(p_n(1), p_n(2), \ldots, p_n(i)\}$ has period of $lcm(1, 2, \ldots, i+1)$ and $t = lcm(1, 2, \ldots, i+1)$ is the smallest value such that $(p_t(1), p_t(2), \ldots, p_t(i)\} = (0, 0, \ldots, 0)$. □

Lemma 1. $p_n(i) - p_n(i-1) = (i-1)(m_n(i) - m_n(i-1)) + 2m_n(i)$

Proof. This is simple to prove.

$$\begin{aligned} p_n(i) - p_n(i-1) &= im_n(i) - (i-1)m_n(i) - b_n(i) + b_n(i-1) \\ &= im_n(i) - (i-1)m_n(i-1) - b_n(i) + (m_n(i) + b_n(i)) \\ &= im_n(i) - (i-1)m_n(i-1) + m_n(i) \\ &= (i-1)(m_n(i) - m_n(i-1)) + 2m_n(i). \end{aligned}$$

□

Lemma 2. *The sequence* $(m_n(i))_{i=1}^{\infty}$ *is non-increasing.*

Proof. This follows from Lemma 1. Since $p_n(i) - p_n(i-1) \leq i$,

$$(i+1)(m_n(i) - m_n(i-1)) + 2m_n(i-1) \leq i.$$

Because $2m_n(i-1) \geq 0$, it follows that

$$(i+1)(m_n(i) - m_n(i-1)) \leq i.$$

Since $m_n(i) - m_n(i-1)$ is an integer, it must be true that $m_n(i) - m_n(i-1) \leq 0$. □

Theorem 6. *The smallest number of stones to require pit n to be used in a winning game is $\frac{n^2}{\pi} + O(n)$.*

Proof. Let $n \in \mathbb{N}^+$ be fixed, and let $f(M)$ be the function that returns the smallest i such that $m_n(i) = M$. By Lemma 2, $m_n(i) = M$ iff $i \in I_M$ where $I_M = \{f(M), f(M+1), \ldots, f(M-1)-1\}$. For any $i \geq 2$ such that $i, i-1 \in I_M$, $p_n(i) - p_n(i-1) = 2M$ by lemma 1. Therefore, the sequence $S_M = (p_n(i))_{i \in I_M}$ is an arithmetic sequence with a difference between terms of $2M$. The other possibility for any $i \geq 2$ is that $i \in I_M$, but $i \notin I_M$. This happens when $i = f(M)$. In this case,

$$p_n(f(M)) - p_n(f(M)-1) = (f(M)-1)(M - m_n(f(M)-1)) + 2M.$$

Because $p_n(f(M)-1) \leq f(M)-1$ and $M - m_n(f(M)-1) \leq -1$, $0 \leq p_n(f(M)) \leq 2M$. Similarly,

$$f(M-1) - 2M + 1 \leq p_n(f(M-1)-1) \leq f(M-1) - 1.$$

This means that

$$\begin{aligned} f(M-1) - f(M) &= \frac{p_n(f(M-1)-1) - p_n(f(M))}{2M} + 1 \\ &= \frac{f(M-1)}{2M} + k. \end{aligned}$$

for some $|k| \leq 3$. It follows that

$$f(M) = \frac{2M-1}{2m} f(M-1) + k.$$

where $|k| \leq 3$. Expanding this out gives

$$\begin{aligned} f(M) = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2M-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2M} x + kM &= \frac{\frac{1}{2} \cdot \frac{3}{\cdot} \cdot \cdots \cdot (2M-1)}{M!} x + kM \\ &= \frac{\Gamma(M + \frac{1}{2})}{m! \sqrt{\pi}} n + kM. \end{aligned}$$

where $|k| \leq 3$ and $x + 1 = M(0)$.

The final step is to compute the value of $s(x) = \sum_{i=1}^{x} p_n i$. This is close to the sum of the sums of the I_M's for high enough x. Each I_M has $f(M-1)/2M+k$ terms for some $|k| \leq 3$ and has an average of $(p_n((M-1)-1)-p_n(f(M)))/2 = f(M-1)/2+kM$ for some $|k| \leq 1$. Multiplying these together gives that each I_M has a sum $f(M-1)^2/4M + O(f(M-1))$. This means that

$$s(x) \sim \sum_{M=1}^{\infty} \frac{\Gamma(M+\frac{1}{2})^2 x^2}{4\pi M!(M-1)!} = \frac{x^2}{4\pi} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; 1\right)$$

by Gauss's summation formula. The result of the theorem follows from the fact that the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 2; 1)$ equals 4. □

5 Applications

Definition 4. *ManQala is a variant of Tchoukaillon made for use in quantum state engineering. The stones in ManQala represent bosonic states, pits represent system modes, and sowing represents unitary operations.*

Due to the quantum nature of ManQala, the outcome of a game is not always the same. Even in the game with only 3 stones, there is a 1 in 3 chance to end up with one stone in the Roumba and 2 in pit 1. Additionally, it is possible for stones to move backwards, which can also occur in the game with 3 stones.

For more information on this topic, see [3].

References

[1] Duane M. Broline and Daniel E. Loeb. The combinatorics of Mancala-type games: Ayo, Tchoukaitlon, and 1/pi, 1995; arXiv:math/9502225.

[2] Brant Jones, Laura Taalman and Anthony Tongen. Solitaire Mancala Games and the Chinese Remainder Theorem, 2011; arXiv:1112.3593.

[3] Onur Danaci, Wenlei Zhang, Robert Coleman, William Djakam, Michaela Amoo, Ryan T. Glasser, Brian T. Kirby, Moussa N'Gom and Thomas A. Searles. ManQala: Game-Inspired Strategies for Quantum State Engineering, 2023; arXiv:2302.14582.