ON CLASSICAL IMPARTIAL GAMES AND THE MATHEMATICS BEHIND THEM

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Abstract. In this paper, we will discuss two classic impartial games, specifically Fibonacci nim and Wythoff's game. We will explore the winning strategies for each, along with exploring the theory underlying them. Such theory includes the Zeckendorf representation of a positive integer and Beatty sequences for Fibonacci nim and Wythoff's game, respectively.

1. Fibonacci nim

We start by exploring Fibonacci nim. First, we explain the rules of the game.

Definition 1.1 (Rules of Fibonacci Nim)**.** The game Fibonacci nim is played between two players, Left and Right (whom we will refer to as "she" and "he", respectively), and it starts with a pile of n tokens; Left goes first. On the first move, she may remove as many tokens as we wish, provided that she does not remove all n tokens. On every proceeding move, the number of tokens removed must not exceed twice the number of tokens removed on the previous move. The player who removes the last token wins.

The reason the game is called Fibonacci nim is because the Fibonacci numbers occur naturally: Right wins going second if and only if n is a Fibonacci number. The Fibonacci numbers occur in our analysis of this game because of the following theorem from Zeckendorf:

Theorem 1.2 (Zeckendorf)**.** *Every positive integer n can be uniquely written of the following form:*

$$n = F_{a_1} + F_{a_2} + F_{a_3} + \cdots + F_{a_k}$$

where $a_1, \ldots, a_n \in \mathbb{N}$ with $a_1 \geq 2$ and $a_i - a_{i-1} \geq 2$ for all $2 \leq i \leq k$.

Before we start the proof of Zeckendorf's theorem, we present the following lemma:

Lemma 1.3. *If $d_1, d_2, \ldots, d_n$ is a positive integer sequence with $d_1 \geq 2$ and $d_i - d_{i-1} \geq 2$ for all $2 \leq i \leq n$, then*

$$\sum_{i=1}^{n} F_{d_i} < F_{d_n+1}.$$

Proof. Fix $k = d_n$; clearly over all sequences d_i with $d_n = k$, the maximum value that the sum can take is

$$F_k + F_{k-2} + F_{k-4} + F_{k-6} + \ldots.$$

We show that this is less than F_{k+1} by inducting on k. The base cases $k = 2$ and $k = 3$ are immediate, since $F_2 < F_3$ and $F_3 < F_4$. For the inductive step, if we assume

$$F_{k-2} + F_{k-4} + \cdots < F_{k-1},$$

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we may add F_k to both sides to get

$$F_k + F_{k-2} + F_{k-4} + \cdots < F_{k-1} + F_k = F_{k+1}.$$

Therefore, the induction is complete, and so

$$F_k + F_{k-2} + F_{k-4} + F_{k-6} + \cdots < F_{k+1}$$

for all $k \in \mathbb{N}$. This concludes our proof of the lemma. ■

Now we will begin our proof of Zeckendorf's theorem.

Proof. The case $n = 1$ is immediate since $1 = F_2$, so assume that $n > 1$. First, we will show that n has a (not necessarily unique) Zeckendorf representation; that is, we will show that we can write n in the above form. We proceed by induction on n; the base case $n = 2$ is clear since $2 = F_3$. For the inductive step, assume that all $1 \le k \le n-1$ have a Zeckendorf representation. Let F_a be the largest Fibonacci number less than or equal to n. If $n = F_a$, we are done since that is the Zeckendorf representation. If $n = F_a + 1$, then we are once again done, since $a \ge 3$ (as $n \ge 2$) and so $a - 1 \ge 2$. Otherwise, $n - F_a \ge 2$. If the greatest Fibonacci number less than or equal to $n - F_a$ is F_{a-1}, then $F_{a-1} \le n - F_a$ implies $n \ge F_{a-1} + F_a = F_{a+1}$, contradicting maximality. Thus if F_b is the largest Fibonacci number less than $n - F_a$, then $a - b \ge 2$. We then apply the inductive hypothesis on $n - F_a$, which proves the existence of a Zeckendorf representation.

Now we will prove that the Zeckendorf representation of a positive integer n is unique. Assume for the sake of contradiction that we can write

$$n = \sum_{a \in A} F_a = \sum_{b \in B} F_b,$$

where $A \neq B$ are sets of positive integers with no consecutive elements. Let $C = A \cap B, A' = A \setminus C, B' = B \setminus C$; we can split each sum as

$$\sum_{a' \in A'} F_{a'} + \sum_{c \in C} F_c = \sum_{b' \in B'} F_{b'} + \sum_{c \in C} F_c,$$

which implies that

$$\sum_{a' \in A'} F_{a'} = \sum_{b' \in B'} F_{b'}.$$

Let k and l be the maximum elements in A' and B', respectively; note that $k \neq l$ since the two sets are disjoint. Without loss of generality assume that $k < l$. By Lemma 1.3, we have that

$$\sum_{a' \in A'} F_{a'} < F_{k+1} \le F_l \le \sum_{b' \in B'} F_{b'}$$

where the second inequality comes from $k < l$. This contradicts the fact that they are equal. Therefore, $A = B$, implying that the Zeckendorf representation is unique, and we are done. ■

Our next few key insights are much simpler and easier to prove than Zeckendorf's theorem. We refer to them as lemmas for ease of reference.

Lemma 1.4. *For all $k \ge 2$, we have $2F_k < F_{k+2}$.*

Proof. A simple calculation yields that $F_{k+2} = F_{k+1} + F_k > F_k + F_k = 2F_k$, thus proving our lemma. ■

Lemma 1.5. *For all $k \geq 3$, we have $\frac{2}{3}F_k \geq F_{k-1}$.*

Proof. This is equivalent to proving that $F_k \geq \frac{3}{2}F_{k-1}$. Writing $F_k = F_{k-1} + F_{k-2}$, we get

$$F_{k-1} + F_{k-2} \geq \frac{3F_{k-1}}{2}.$$

Rearranging and once again using the definition gives

$$F_{k-2} \geq \frac{F_{k-1}}{2} = \frac{F_{k-2} + F_{k-3}}{2}.$$

Once again rearranging, it suffices to show $F_{k-2} \geq F_{k-3}$, which is clearly true. ∎

We will now begin our proof that Right wins going second if and only if n is a Fibonacci number. In fact, we will show an even stronger statement:

Theorem 1.6. *Suppose that Left and Right are playing in a position (n, q) of Fibonacci nim where there are n tokens in the pile, and Left to move can remove at most q tokens on her move. Then Left wins if and only if when we write n's Zeckendorf representation*

$$n = \sum_{i=1}^{k} F_{a_i},$$

we have that $F_{d_1} \leq q$. Otherwise, Right wins. Call such a position (n, q) good *if it satisfies the property above and* bad *otherwise.*

Proof. First, we will show that every good position has a move to a bad position. This is not hard; by definition $F_{a_1} \leq q$, so Left to move may remove F_{a_1} tokens, leaving

$$\sum_{i=2}^{\infty} F_{a_i}$$

tokens remaining. Now, Right to move can remove at most $q' = 2F_{a_1} < F_{a_2}$ tokens by Lemma 1.3. This means that our new position is bad, as needed.

Now we will show that every bad position only has moves to a good position. Suppose that Left to move removes $x < F_{a_1}$ tokens. It suffices to show that in the Zeckendorf representation of $F_{a_1} - x$, the smallest term is at most $2x$. We prove this by induction on a_1; the base cases $a_1 = 3$ and $a_1 = 4$ are immediate, since $2 - 1 = 1 \leq 2$, $3 - 1 = 2 \leq 2$, and $3 - 2 = 1 \leq 4$. For the inductive step, if $x \geq \frac{F_{a_1}}{3}$ then the smallest term is less than $F_{a_1} - x \leq 3x - x = 2x$, as needed, so assume that $x < \frac{F_{a_1}}{3}$. Then by Lemma 1.4, we get

$$x - \frac{F_{a_1}}{3} > F_{a_1} - \frac{F_{a_1}}{3} = \frac{2F_{a_1}}{3} > F_{k-1}.$$

This means by the algorithm described in Theorem 1.1, the number F_{k-1} is in the Zeckendorf representation of $F_{a_1} - x$. Therefore, it suffices to show that the smallest term in the Zeckendorf representation of $F_{a_1} - x - F_{a_1 - 1} = F_{a_1 - 2} - x$ is at most $2x$, which follows by the inductive hypothesis. This implies that every bad position has moves only to a good position.

Finally, every good position can move to a bad position, from which a bad position only has moves to a good position. This implies that all good positions are winning for Left, whereas all bad positions are winning for Right, as desired. ∎

Now, the setup of normal Fibonacci nim uses $q = n-1$. If n is not a Fibonacci number, then the smallest number in its Zeckendorf representation is by definition less than n, and thus less than or equal to $n-1$, making it a winning position for Left. If n is a Fibonacci number on the other hand, then the smallest number in its Zeckendorf representation is equal to n, which is greater than $n-1$. This makes n a winning position for Right. This implies that Right wins if and only if n is a Fibonacci number, which concludes our study of Fibonacci nim.

2. Wythoff's game

We shall now begin our discussion of Wythoff's game. First, we explain the rules of the game. As before, the game is between two players, Left and Right, with Left moving first. At the start of the game, there are two piles, one with m tokens and another with b tokens. On each move, the current player can either remove as many tokens as they would like from either pile, or they can remove tokens from both piles, so long as they remove the same number of tokens from each pile. For example, the valid moves from the position $(2, 3)$ are to

$$(1, 3), (0, 3), (2, 2), (2, 1), (2, 0), (1, 2), (0, 1).$$

The player who has no valid moves left loses; thus a player wins if they are the first to clear both piles.

The winning positions are described below:

Theorem 2.1. *Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. A position (a, b) is a winning position for Right in Wythoff's game if (a, b) is of the form $(\lfloor k\phi \rfloor, \lfloor k\phi^2 \rfloor)$ or $(\lfloor k\phi^2 \rfloor, \lfloor k\phi \rfloor)$, where k is a nonnegative integer. Otherwise, (a, b) is a winning position for Left. Call positions of the above form bad, and call all other positions good.*

The reason why the golden ratio comes into play is because of the following theorem from Rayleigh on Beatty sequences:

Theorem 2.2 (Rayleigh)**.** *Let r, s be irrational numbers with $r, s > 1$. with $\frac{1}{r} + \frac{1}{s} = 1$. Let $a_k = \lfloor kr \rfloor$ and $b_k = \lfloor ks \rfloor$ for all positive integers k. Then every positive integer n is contained in exactly one of the sequences $\{a_k\}_{k=1}^{\infty}$ or $\{b_k\}_{k=1}^{\infty}$.*

Proof. First, we will show that no positive integer can be in both sequences. Assume for the sake of contradiction that there exist positive integers n, m, k such that $\lfloor rk \rfloor = n$ and $\lfloor sm \rfloor = n$. Note that $n \neq rk$ and $n \neq sm$, otherwise r and s are rational. Then

$$rk < n < rk + 1$$

and

$$sm < n < sm + 1.$$

Dividing the first relation by r and the second by s, we get

$$k < \frac{n}{r} < k + \frac{1}{r}$$

and

$$m < \frac{n}{s} < m + \frac{1}{s}.$$

Adding the two and using the fact that $\frac{1}{r} + \frac{1}{s} = 1$ gives us

$$k + m < n < k + m + 1.$$

This implies that an integer is between two consecutive integers, which is a contradiction. Therefore, no positive integer can be in both sequences.

Now we will show that there is no positive integer n such that n is in neither sequence. Assume for the sake of contradiction that there exists such an n. The sequence $\{a_i\}_{i=1}^{\infty}$ divides the positive real numbers into intervals of the form $[kr, kr+1)$ for positive integers k, and a positive integer n is in the sequence if and only if it lies in one of these intervals. Thus if n is not in the sequence, it must be in an interval between the two intervals $[kr, kr+1)$ and $[(k+1)r, (k+1)r+1)$. This implies that for some positive integer k, we have that

$$kr + 1 \leq n < kr + r.$$

The first inequality is actually strict because r is irrational, so

$$kr + 1 < n < kr + r.$$

Similarly, for some positive integer m, we have that $ms + 1 < n < ms + s$. Dividing the first relation by r and the second relation by s, we get

$$k + \frac{1}{r} < \frac{n}{r} < k + 1$$

and

$$m + \frac{1}{s} < \frac{n}{s} < m + 1.$$

Adding the two and using the fact that $\frac{1}{r} + \frac{1}{s} = 1$ gives us

$$k + m + 1 < n < k + m + 2.$$

As before, this implies that an integer is between two consecutive integers, which is a contradiction. Thus no positive integer is in neither sequence.

Therefore, every positive integer is in exactly one of the sequences $\{a_k\}_{k=1}^{\infty}$ or $\{b_k\}_{k=1}^{\infty}$, as desired. ■

For simplicity, let $c_k = \lfloor k\phi \rfloor$ and $d_k = \lfloor k\phi^2 \rfloor$. A well-known property of ϕ is that

$$\frac{1}{\phi} + \frac{1}{\phi^2} = 1;$$

this may be verified either by repeatedly applying the identity $\phi^2 = \phi + 1$ or by direct expansion. By Theorem 2.2, this implies that every positive integer is in exactly one of the sequences $\{c_k\}_{k=1}^{\infty}$ or $\{d_k\}_{k=1}^{\infty}$.

Now we will begin our proof of Theorem 2.1.

Proof. First, we must show that every bad position only has moves to a good position. Assume that Left to move is at the position (a, b) with $a = c_k$ and $b = d_k$; similar analysis applies when Left to move is at the position (b, a). On her move, she can either decrease a to any nonnegative integer less than it, decrease b to any nonnegative integer less than it, or decrease a and b by the same amount, so long as the results are both nonnegative integers. In the first case, note that $a = c_k$, so $a \neq c_i$ for $i \neq k$ and $a \neq d_j$ for all j. This means that decreasing a will yield a good position. Similarly, decreasing b will yield a good position. Finally, if we decrease both a and b, we do not change $b - a$. However,

$$d_k - c_k = \lfloor k\phi^2 \rfloor - \lfloor k\phi \rfloor = \lfloor k\phi + k \rfloor - \lfloor k\phi \rfloor = k,$$

so $c_k - d_k = -k$. Therefore, if she goes from (a, b) to a bad position while keeping $a - b$ constant, she must stay at (a, b), which is against the rules. Thus in any case, every bad position only has moves to a good position.

Now we must show that every good position (a, b) has some move to a bad position. Without loss of generality, assume that $a < b$; if $a = b$ we just move to $(0, 0)$, which is bad. There are two cases:

Case 1: $a = c_k$ for some k. Then if $b > d_k$, we can move from (a, b) to (c_k, d_k), which is a bad position. Otherwise, $a \leq b < d_k$. Let $m = b - a$, so that $d_m - c_m = m = b - a$. We claim that $k < b - a$, so that $a < c_m$, which means that there is a move from (a, b) to (c_m, d_m). Indeed, this reduces to $k < b - c_k$, which is indeed $b > d_k$.

Case 2: $a \neq c_k$ for all k. Then $a = d_m$ for some m, implying $c_m \leq d_m = a < b$, so there is a move from (a, b) to (d_m, c_m).

Therefore, every good position has a move to a bad position, which combined with our earlier work implies that all good positions are winning positions for Left, and all bad positions are winning for Right, as desired. ■

Example. Let's explore our winning strategy in Wythoff's game with $(a, b) = (15, 19)$. The first few terms of the sequences c_i and d_i are

$$0, 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19$$

and

$$0, 2, 5, 7, 10, 13, 15, 18.$$

Then Left to move notices that by Theorem 2.1, the position $(15, 9)$ is bad and thus winning for the second player, so she moves there. Right then has no moves to a good position on his move, so suppose he moves to $(11, 5)$. Left to move now sees that $(3, 5)$ is a winning position for the second player, so she moves there. Right then must move to a good position for Left, say $(3, 1)$. Left now moves to $(2, 1)$, from which Right either moves to $(2, 0), (0, 1), (1, 1)$, or $(1, 0)$. All of these positions are bad, so Left wins here.

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