CLASSICAL IMPARTIAL GAMES

SUNAY MIDUTHURI

Abstract. Impartial games are games in which both players have the same options. Aside from Nim, two of the most famous impartial games are Fibonacci Nim and Wythoff's Game. In Fibonacci Nim, players take stones from a pile such that the player to move may take at most twice the number of stones removed on the previous turn. In Wythoff's Game, there are two piles, and the player to move may take any amount of stones from either or the same number of stones from both. In both games, the first player who cannot make a move loses. When considering these two games, the natural question is who wins. This paper will answer these questions and dive deep into the number-theoretical foundations of these games.

1. Introduction

In many common games, the players have different moves available to them. For example, in chess, one player moves the black pieces while the other moves the white pieces. Games like this, where both players have completely different moves, are called *partizan*. On the other hand, other games that have the same set of moves for both players are called *impartial*. The set of possible moves is called the set of *options* of the game. This paper focuses on these impartial games, and we'll begin with the most common one: nim.

2. Nim

Definition 2.1. *In the game of* nim, *the two players take turns moving. The game starts with k piles of stones $p_1, p_2, \ldots, p_k$ such that the ith pile has size s_i. On a player's turn, they pick one pile p_j and remove any positive number of stones from it. The player who can no longer move loses.*

So what positions are winning and losing?

Definition 2.2. *An $\mathcal{N}$ position is one in which the Next player to move wins. A $\mathcal{P}$ position is one in which the next player loses so that the Previous player is the winner.*

Theorem 2.3 (Partition Theorem)**.** *Consider a set $\mathcal{J}$ of short impartial games such that for all $G \in \mathcal{J}$, all subpositions of G are in $\mathcal{J}$. Suppose there exist disjoint subsets $\mathscr{P}$ and $\mathscr{N}$ that partition $\mathcal{J}$ such that for all games $G \in \mathscr{P}$, all options of G are in $\mathscr{N}$, and for all games $G \in \mathscr{N}$, there exists an option of G in $\mathscr{P}$. Then $\mathscr{P}$ and $\mathscr{N}$ are the $\mathcal{P}$ and $\mathcal{N}$ positions of $\mathcal{J}$, respectively.*

Proof. Intuitively, this theorem seems to make sense; in an $\mathcal{N}$ position, where the player to move must win, there should be a way for them to move to a position in which they move, which is now a $\mathcal{P}$ position. On the other hand, in a $\mathcal{P}$ position, the player to move should lose no matter what they play, so all of their moves must be to an $\mathcal{N}$ position. So clearly these sets satisfy the condition.

We'll induct on the birthday of G to show that they are the only such sets. If the birthday of G is 0, then $G \in \mathcal{P}$. In such a partition, since there doesn't exist an option of 0 in $\mathscr{P}$, 0 must be sorted into the set $\mathscr{P}$, which is indeed the correct sorting. Now, assume that all games with birthdays less than n are sorted correctly. Now consider a game G with birthday n. If $G \in \mathscr{P}$, we know that all options of G are in $\mathscr{N}$. But since options of G have birthday $n-1$, our inductive hypothesis tells us that they are $\mathcal{N}$ positions. Thus G is a $\mathcal{P}$ position. Similarly, we find when $G \in \mathscr{N}$ that G is an $\mathcal{N}$ position, so the desired result follows by induction. $\square$

Next, we'll define the operation at the heart of our analysis of nim.

Definition 2.4. *Let $\oplus$, or the nim sum (called "xor" in computer science), be an operation such that $a \oplus b$ adds the two integers in binary without carrying.*

Let's try evaluating $10 \oplus 14$. We get $10_{10} = 1010_2$ and $14_{10} = 1110_2$. Adding without carrying, we obtain the following:

$$\begin{array}{r} 1010 \\ \oplus\ 1110 \\ \hline 0100 \end{array}.$$

Thus, $10 \oplus 14 = 100_2 = 4$. Before we use the nim sum, we need a few lemmas to help us understand it.

Lemma 2.5. *For all positive integers n, $n \oplus n = 0$.*

Proof. Suppose that when written in binary,

$$n = \sum_{i=0}^{k} a_i 2^i,$$

where $a_i \in \{0, 1\}$. When computing the nim sum $n \oplus n$, we go through each digit from right to left and find the corresponding digit in the nim sum. For each index i, a_i is the digit in both rows of the binary addition. Thus, the ith digit from the right of $n \oplus n$ is 0, so $n \oplus n = 0$, as desired. □

Lemma 2.6. *The nim sum is commutative and associative.*

Proof. Each digit of the nim sum is the result of a sum of digits in each number. If we permute the numbers, the nim sum will therefore remain the same, so the nim sum is indeed associative and commutative by the corresponding properties of normal addition. □

Lemma 2.7. *Given n integers $a_1, a_2, \ldots, a_n$, there exists a unique integer a_{n+1} such that*

$$a_1 \oplus a_2 \oplus \cdots \oplus a_{n+1} = 0.$$

Proof. Let $x = a_1 \oplus a_2 \oplus \cdots \oplus a_n$. Then x has some digits that are 0 and some that are 1. By Lemma 2.6, the desired condition is equivalent to proving that there is a unique integer a_{n+1} such that $a_{n+1} \oplus x = 0$. Let

$$x = \sum_{i=0}^{k} b_i 2^i,$$

where $b_i \in \{0, 1\}$, and let

$$a_{n+1} = \sum_{i=0}^{j} c_i x^i.$$

For each digit i, if $b_i = 1$, c_i must be 1 for the ith digit of the nim sum to be 0. Similarly, if $b_i = 0$, $c_i = 0$ as well. Thus we find $a_{n+1} = x$ is the unique integer satisfying $a_1 \oplus a_2 \oplus \cdots \oplus a_{n+1} = 0$. □

The next theorem, from Bouton [Bou02], shows the significance of the nim sum to the game.

Theorem 2.8. *The position with piles of size $s_1, \ldots, s_k$ is a P position if and only if $s_1 \oplus s_2 \oplus \cdots \oplus s_k = 0$.*

Proof. Let $\mathscr{P}$ be the set of positions $\{s_1, \ldots, s_k\}$ such that $s_1 \oplus s_2 \oplus \cdots \oplus s_k = 0$, and let $\mathscr{N}$ be the set of positions such that the nim sum is not zero. Clearly, these two are disjoint and partition the set of NIM positions.

Let's begin with a position in $\mathscr{N}$, so the nim sum is nonzero. We must show that there is a move we can make to make the nim sum zero. Suppose this nim sum is

$$S = s_1 \oplus s_2 \oplus \cdots \oplus s_k.$$

We can write this in binary, so that

$$S = \sum_{i=0}^{r} a_i 2^i,$$

where $a_i \in \{0, 1\}$. We have that a_r is the leading digit. Once we clear out the zeros from the front - note that since $S \neq 0$ there must be a 1 somewhere - let l be the index of that 1. Then $a_l = 1$, so by the definition of the nim sum there must be an odd number of s_i with a 1 in the 2^l position. Pick one of these, s_x. Then s_x and S both have a 1 in the 2^l position. This means that $s_x \oplus S$ has a 0 in that position, so $s_x \oplus S < s_x$.

Consider the move $\mathscr{M}$ that takes away $s_x - (s_x \oplus S)$ stones from the pile with s_x stones, leaving $s_x \oplus S$ stones. The nim sum of the resulting position is

$$\begin{aligned} S' &= s_1 \oplus s_2 \oplus \cdots \oplus s_{x-1} \oplus (s_x \oplus S) \oplus s_{x+1} \oplus \cdots \oplus s_k \\ &= (s_1 \oplus \cdots \oplus s_k) \oplus S \\ &= S \oplus S \\ &= 0, \end{aligned}$$

so we move to a position in $\mathscr{P}$.

Next, we consider a position in $\mathscr{P}$ with nim sum 0. We aim to show that all moves from this position are in $\mathscr{N}$. Suppose we move in pile p. Since the nim sum is 0, we know that

$$s_p = s_1 \oplus \cdots \oplus s_{p-1} \oplus s_{p+1} \oplus \cdots \oplus s_k.$$

By Lemma 2.7, s_p is the unique value such that the nim sum of all the piles is 0. Therefore, moving in pile p will decrease s_p and therefore change the nim sum into a nonzero value. Thus all moves from $\mathscr{P}$ are in $\mathscr{N}$.

Therefore, by the Partition Theorem, we arrive at the desired result. □

3. Nimbers and the Mex Rule

Definition 3.1. *We define the nimbers*

$$*n = \{*0, *1, \ldots, *(n-1)\},$$

or the game whose options are all the nimbers before it. We abbreviate $*0 = 0$ *and* $*1 = *$.

Observe that $*n$ is simply the value of a NIM pile with size n.

Theorem 3.2. $*a_1 + *a_2 + \cdots + *a_k = *(a_1 \oplus a_2 \cdots \oplus a_k)$.

Proof. We know that $G = H$ for two games G and H if and only if $G - H \in \mathcal{P}$. For impartial games $H = -H$, so this is equivalent to showing that $G + H \in \mathcal{P}$. The desired result then follows immediately from Lemma 2.7. □

Definition 3.3. *Let S be a subset of the nonnegative integers. Then* mex (S) *(Minimum EXcludant) is the minimal nonnegative integer not contained in S.*

The mex is also important to our study of nimbers.

Theorem 3.4 (Mex Rule)**.** *The game $G = \{a_1, a_2, \ldots, a_n\}$ is equivalent to $*m$, where $m = mex\ (a_1, \ldots, a_n)$.*

Proof. We aim to prove that $G + *m \in \mathcal{P}$. The first player can either move in G or $*m$. If they move in G, say to a_i, we are left with $a_i + m$. By definition, $m \neq a_i$, so the second player moves a_i to m if $a_i > m$, and vice versa otherwise. We are now left with $*m, *m$ or $*a_i, *a_i$, which by Lemma 2.5 and Theorem 2.8 is a $\mathcal{P}$-position. On the other hand, if they move in $*m$ to $*a$, $*a \in \{a_i\}$ by the definition of m, so the move from G to $*a$ wins for the second player. Thus $G + *m \in \mathcal{P}$, as desired. □

4. Sprague-Grundy Theory

Definition 4.1. *If G is an impartial game such that $G = *n$, we define the Grundy Value $\mathcal{G}(G)$ of G to be equal to n.*

The following theorem, discovered by Sprague [Spr36] and Grundy [Gru39], is the central idea for the theory of classical impartial games.

Theorem 4.2. *Any short impartial game G can be expressed as $*n$ for some nonnegative integer.*

Proof. We'll induct on the birthday of G. The base case 0 is trivial. Now assume that the theorem is true for the options of G, so that each option can be expressed as $*a_k$. Then

$$G = \{*a_1, *a_2, \ldots, *a_k\} = *m,$$

where $m = \text{mex}\ (a_1, \ldots, a_k)$. By induction, we arrive at the desired result. □

Given these tools, we move on to the two most common variants of Nim that we'll be analyzing.

5. Fibonacci Nim

FIBONACCI NIM [Whi63] is a variant of Nim with extremely important connections to Number Theory.

Definition 5.1. *In* FIBONACCI NIM *, players take stones from a single pile. However, each player may only take at most twice the number of stones taken on the previous turn. (The first player may take as many stones as desired, but not the whole pile.)*

As usual, the game is lost when a player has no moves left. This can happen at 0, but this game can also be lost in the opening position 1. We denote a position with (n, r) if there are n stones in the pile and the maximum number that can be removed is r.

Definition 5.2. *The Fibonacci numbers F_n satisfy $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.*

Lemma 5.3. *Given an increasing sequence $a_1, a_2, \dots, a_k$ such that $a_{i+1} > a_i + 1$ and $a_1 > 1$,*

$$\sum_{i=1}^{k} F_{a_i} < F_{a_k+1}.$$

Proof. We'll induct on k. If $k = 1$, the problem asks us to prove that $F_{a_1} < F_{a_1+1}$, which follows simply from the fact that the Fibonaccis are increasing. Now, suppose that $k > 1$. By the inductive hypothesis, we know that

$$\sum_{i=1}^{k-1} F_{a_i} < F_{a_{k-1}+1}.$$

This means that

$$\sum_{i=1}^{k} F_{a_i} = \sum_{i=1}^{k-1} F_{a_i} + F_{a_k} < F_{a_{k-1}+1} + F_{a_k} \leq F_{a_k-1} + F_{a_k} = F_{a_k+1},$$

where the second inequality follows from the fact that $a_{k-1} + 1 < a_k$. □

Through this Lemma, we can prove the following important result: [Hen16]

Theorem 5.4 (Zeckendorf's Theorem)**.** *Every positive integer can be uniquely represented as a sum of distinct, nonconsecutive Fibonacci numbers.*

Proof. We'll start by showing the existence of such a representation through induction. For our base case, 1 is already a Fibonacci number. Now, let $n > 1$. If n is Fibonacci, then we're done. Otherwise, there is a positive integer a such that $F_a < n < F_{a+1}$. By the inductive hypothesis, this means that $x = n - F_a$ has a Zeckendorf representation. We have that $F_a + x = n < F_{a+1} = F_a + F_{a-1}$, or $x < F_{a-1}$. Then every Fibonacci in the Zeckendorf Representation is at most F_{a-2}, so we can simply add F_a to the representation to obtain a valid Zeckendorf representation of n.

Now, we must show uniqueness. We'll use contradiction; let n be the smallest number with at least two Zeckendorf representations. We can write $F_{a_1} + F_{a_2} + \cdots + F_{a_j} = n = F_{b_1} + F_{b_2} + \cdots + F_{b_k}$. If any of the a_i equaled one of the b_i, we could remove these terms to get a smaller n, which is impossible; therefore, the a_i and b_i are distinct. WLOG assume $a_1 < a_2 < \cdots < a_j$, $b_1 < b_2 < \cdots < b_k$, and $a_j < b_k$. Then

$$n = \sum_{i=1}^{j} F_{a_i} < F_{a_j+1} \leq F_{b_k} \leq \sum_{i=1}^{k} F_{b_i}$$

by Lemma 5.3, a contradiction, so the desired result follows immediately. □

Definition 5.5. *We let $z_i(n)$ denote the ith smallest part of the Zeckendorf Representation of n.*

Lemma 5.6. *Let $n > 1$ and $1 \leq k < z_1(n)$ so that $z_1(k) = F_t$. Then $z_1(n-k) \in \{F_{t-1}, F_{t+1}\}$.*

Proof. We use induction on the number z of Zeckendorf parts of k. If $z = 1$, k is a Fibonacci number and $k = F_t$. Let $z_1(n) = F_s$. If $s \equiv t \pmod 2$, we have that $s = t + 2d$. Therefore,

$$\begin{aligned} F_s - k &= F_s - F_t \\ &= F_s - F_{s-2d} \\ &= (F_s - F_{s-2}) + (F_{s-2} - F_{s-4}) + \cdots + (F_{s-2d+2} - F_{s-2d}) \\ &= F_{s-1} + F_{s-3} + \cdots + F_{s-2d+1}, \end{aligned}$$

meaning that

$$z_1(F_s - k) = z_1(n - k) = F_{s-2d+1} = F_{t+1}.$$

If $s \not\equiv t \pmod 2$, we have $t = s - 2d - 1$, and find similarly that

$$F_s - k = F_{s-1} + \cdots + F_{s-2d+1} + F_{s-2d-2},$$

so

$$z_1(F_s - k) = z_1(n - k) = F_{s-2d-2} = F_{t-1}.$$

Therefore, our base case is satisfied. Next, suppose $z > 1$ and $z_1(k) = F_t$. Then $z_1(k - F_t) \geq F_{t+2}$. In addition, $k - F_t$ has at most $z - 1$ parts. Then, by the inductive hypothesis, $z_1(n - k + F_t) \in \{F_{t+1}, F_{t+3}$, so

$$z_1(n - k + F_t) > F_{t+1} > F_t = z_1(k).$$

Applying the base case with $n' = n - k + F_t$ and $k' = z_1(k)$, we arrive at the desired result. □

Theorem 5.7. *The* FIBONACCI NIM *position* (n, r) *is a* $\mathcal{P}$ *position if and only if* $r < z_1(n)$.

Proof. Let $\mathscr{P}$ be the set of positions (n, r) where $r < z_1(n)$, and let $\mathscr{N}$ be the rest of positions. Consider $(n, r) \in \mathscr{N}$. Since $r \geq z_1(n)$, we can move to $(n - z_1(n), 2z_1(n))$. If $n = z_1(n)$, this move is clearly to a $\mathscr{P}$ position. Otherwise, if $F_t = z_1(n)$, we have that $z_1(n - z_1(n)) = z_2(n) \geq F_{t+2} = F_{t+1} + F_t \geq 2F_t$, so this is a move to $\mathscr{P}$. Now consider $(n, r) \in \mathscr{P}$. Then $r < z_1(n)$. For all possible moves that take away k stones, $k \leq r < z_1(n)$, so by Lemma 5.6 we can write

$$z_1(n - k) \leq F_{t+1} = F_t + F_{t-1} < 2F_t < 2k,$$

where $F_t = z_1(k)$. Thus $(n - k, 2k) \in \mathscr{N}$, as desired. □

From this formula, we see that the winning move in an $\mathcal{N}$ position (n, r) must be the move to $(n - z_1(n), 2z_1(n))$. This is the move that takes away $z_1(n)$ stones from the pile with size n; in other words, this move removes the smallest element in the Zeckendorf Representation of n.

6. Grundy Values for Fibonacci Nim

While there is no known closed form for the Grundy Values of Fibonacci Nim, [LRS16] finds a form for the Grundy Value in some cases, assuming that $z_i(n) = \infty$ if the number of Zeckendorf parts of n is less than i. The authors show the following theorems:

Theorem 6.1. $\mathcal{G}(n, r) = 1$ *iff* $z_1(n) = 1$ *and* $r \in [1, z_2(n))$.

Theorem 6.2. $\mathcal{G}(n, r) = 2$ *iff* $z_1(n) = 2$ *and* $r \in [2, z_2(n))$.

Theorem 6.3. $\mathcal{G}(n, r) = 3$ *iff* $z_1(n) = 1, z_2(n) = 3$, *and* $3 \leq r < z_3(n)$, *or* $z_1(n) = 3$ *and* $3 \leq r < z_2(n) - 1$.

Theorem 6.4. *For all* $n \geq 0$, *we have* $\mathcal{G}(n, n) \leq \mathcal{G}(n + 1, n + 1) \leq \mathcal{G}(n, n) + 1$.

7. Wythoff's Game

WYTHOFF'S GAME or WYTHOFF NIM is another game with $\mathcal{P}$ positions relating to important number theoretic sequences.

Definition 7.1. WYTHOFF'S GAME is played with two piles. A turn consists of taking some number of stones from a single pile or the same number of stones from both piles.

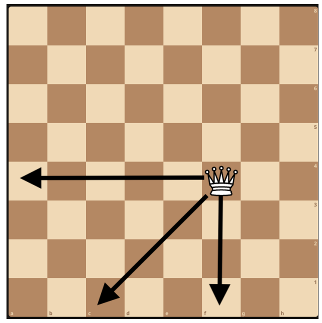

Figure 1. A queen on an 8×8 chessboard

Equivalently, we can consider a queen on a chessboard - the diagram above shows the case of an 8×8 board. As depicted, the queen may move horizontally, vertically, or diagonally. If we consider the number of rows and columns to be the sizes of the two piles, this means that the queen's horizontal or vertical movements consist of taking stones away from a single pile, while diagonal movements take the same number of stones away from each pile. We define a WYTHOFF'S GAME position to be the pair (a, b) corresponding to the sizes of the piles, so that a represents the horizontal distance of the queen while b represents the vertical distance. Figure 1 is therefore $(5, 3)$.

Theorem 7.2. *The $\mathcal{P}$ positions of* WYTHOFF'S GAME *are the pairs (a_n, b_n) so that $a_n = mex\ (a_i, b_i : i \in \{0, 1, \ldots, n-1\})$ and $b_n = a_n + n$.*

Proof. Call the set of positions described $\mathscr{P}$ and the rest $\mathscr{N}$. Then suppose for the sake of contradiction that there exists a move from (a_n, b_n) to (a_m, b_m). Since the minimum of all the coordinates cannot increase, we must have $m < n$ by construction of a_i. By this construction, $a_n \neq a_m$ and $b_n \neq b_m$, so this move must be diagonal. But this is also impossible; for it to be diagonal, we would need $a_n - b_n = a_m - b_m$, or $-n = -m$, which is impossible. Next, we must show that there exists a move from $\mathscr{N}$ to $\mathscr{P}$. Suppose that (a, b) is not of the form (a_i, b_i), and WLOG $a \leq b$. If a is of the form a_i, we must have $b \neq b_i$. If $b > b_i$, there is a move from (a, b) to $(a, b_i) = (a_i, b_i)$, which is a move to a $\mathscr{P}$ position. Otherwise, if $a \leq b < b_i$, let $m = b - a$, so that $m < i \implies a_m < a_i$, and there is a move to (a_m, b_m). Otherwise, if a is not of the form a_i, we must have $a = b_n$ for some n by construction of a_i. Thus $a_n < a$, so there is a move to (b_n, a), completing the proof by symmetry. □

Definition 7.3. *Two sets are complementary iff they partition the positive integers.*

Definition 7.4. *Let α be a positive irrational number. The sequence $a_n = \lfloor \alpha n \rfloor$ is called a Beatty sequence.*

The following theorem from [Bea26] is critical to our understanding of WYTHOFF'S GAME.

Theorem 7.5. *Let α and β be positive irrational numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the Beatty Sequences for α and β are complementary.*

Proof. For some integer N, there are $\lfloor \frac{N}{\alpha} \rfloor$ terms of the Beatty Sequence for α that are less than N, and $\lfloor \frac{N}{\beta} \rfloor$ terms of the Beatty Sequence for β less than N. We know that

$$\frac{N}{\alpha} - 1 < \lfloor \frac{N}{\alpha} \rfloor < \frac{N}{\alpha} \tag{7.1}$$

$$\frac{N}{\beta} - 1 < \lfloor \frac{N}{\beta} \rfloor < \frac{N}{\beta}. \tag{7.2}$$

Adding (7.1) and (7.2) gives $N-2 < \lfloor \frac{N}{\alpha} \rfloor + \lfloor \frac{N}{\beta} \rfloor < N$. Thus $\lfloor \frac{N}{\alpha} \rfloor + \lfloor \frac{N}{\beta} \rfloor = N-1$. In addition, $\lfloor \frac{N+1}{\alpha} \rfloor + \lfloor \frac{N+1}{\beta} \rfloor = N$. The extra 1 must be added to exactly one of the two terms, so the desired result follows immediately. □

This gives us our final theorem:

Theorem 7.6. *The $\mathcal{P}$ positions of* WYTHOFF'S GAME *are the positions (a, b) and (b, a) such that*

$$a = \lfloor \phi n \rfloor \text{ and } b = \lfloor \phi^2 n \rfloor.$$

Proof. Note that $\phi^2 = \phi + 1$, meaning that $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$. Then the Beatty Sequences for ϕ and ϕ^2 are complementary. In addition,

$$\lfloor n\phi^2 \rfloor = \lfloor n\phi + n \rfloor = \lfloor n\phi \rfloor + n.$$

This means that $\lfloor n\phi \rfloor = \text{mex}\left(\lfloor m\phi \rfloor, \lfloor m\phi^2 \rfloor : m < n\right)$ and $\lfloor n\phi^2 \rfloor = \lfloor n\phi \rfloor + n$, so the desired result follows from Lemma 4.2. □

While we do know the winning strategy, the Grundy values for WYTHOFF'S GAME are elusive, and no significant patterns have been found.

References

[Bea26] Samuel Beatty. Problem 3173. *The American Mathematical Monthly*, 33(3):159, 1926.

[Bou02] Charles L. Bouton. Nim, a game with a complete mathematical theory. *Annals of Mathematics*, 3(1/4):35–39, 1901/02.

[Gru39] Patrick M. Grundy. Mathematics and games. *Eureka*, 2(6-8):21, 1939.

[Hen16] Nik Henderson. What is zeckendorf's theorem. *OSU*, pages 2–3, 2016.

[LRS16] Urban Larsson and Simon Rubinstein-Salzedo. Grundy values of fibonacci nim. *International Journal of Game Theory*, 45(3):617–625, 2016.

[Spr36] Richard Sprague. Uber mathematische kampfspiele. *Tôhoku Math*, 41:438–444, 1936.

[Whi63] Michael J. Whinihan. Fibonacci nim. *Fibonacci Quarterly*, 1:9–13, 1963.