SOME PROPERTIES OF THE SURREAL NUMBERS

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ABSTRACT. The surreal numbers are a generalization of the real, hyperreals, superreals, and other similar systems. This papers assumes knowledge that the surreal numbers form an ordered field. We define the ω map and use it to derive the Conway normal form for all surreal numbers. Then we define omnific integers, the appropriate analogue of integers for surreal numbers. We close by proving the surreal numbers are real closed, which means that the surcomplex numbers ae algebraically closed.

1. Conway Normal Form

Any real number x can be expressed uniquely as a base-b infinite sum with digits between 0 and 9. Does a similar expansion exist for surreal numbers? There does. Conway normal form is like a base-ω expansion for a surreal number.

Theorem 1.1. *Any surreal number x can be expressed uniquely as a sum of the form*

$$x = \sum_{\alpha<\beta} r_\alpha \omega^{y_\alpha}, \tag{1.1}$$

where β is an ordinal, y_α is any surreal, and r_α is a real number for all α.

Right now, this theorem has a few problems. First, we don't have a definition for ω^x when x is not an integer. We also haven't defined ordinal-indexed sums.

Let's try to define ω^x first. We want ω^x to be positive, so it can't hurt to make 0 a left option. For any left option x^L, we want $r\omega^{x_L}$ to be smaller than ω^x for all positive real numbers r, so we make $r\omega^{x^L}$ a left option of ω^x. Similarly, we want ω^{x_R} to be a right option. It turns out this is a pretty good definition. The reason we need to add 0 as a left option so that $\omega^0 = 1$. Thus we have gotten this definition for exponents.

Definition 1.2 (ω map)**.** Let x be a number. We define

$$\omega^x = \{0, r\omega^{x_L} | r\omega^{x_R}\}, \tag{1.2}$$

where r ranges over all positive reals.

There is another commonly used definition of exponential for any surreal number, not just ω, that doesn't necessarily match our definition, which is why this is often called the ω map. Of course, we want the ω map to behave like an exponential.

Theorem 1.3. *As we have defined it, ω^x behaves like a normal exponential. Specifically,*

$$\omega^x \omega^y = \omega^{x+y}, \omega^0 = 1 \tag{1.3}$$

Date: December 2024.

The proof is just computation. All other properties of exponents follow from this, so our definition is decent.

A lot of times, it can be helpful to know if two numbers are the same "order of magnitude". We formalize this below.

Definition 1.4. We say positive numbers x and y are commensurate if there exists some positive reals a, b such that $ax < y < bx$. If $0 < x < y$, and x and y are not commensurate, we say that x is infinitesimal compared to y, and write $x << y$.

It is easy to prove that commensurativity is an equivalence relation. Also, if $x < z < y$, and x is commensurate to y, then z is commensurate to x and y, which means that these equivalence classes are convex. It is a bit harder to prove every surreal number is in one of these equivalence classes.

Theorem 1.5. *Any positive number x is commensurate with a unique ω^y.*

Proof. We can say by induction that x^L is commensurate to ω^{y_L} and that ω^R is commensurate to ω^{y^R}. We can assume x is not commensurate with any of its options. Since x is not commensurate to ω^{y_L} and $x^L < x$, we have that $r\omega^{y_L} < x$ for any positive real r. Thus we can add $r\omega^{y_L}$ as a left option. Since $x^L < b\omega^{y_L}$ for some b, we can remove x^L as an option without changing the value of x. We can do the same thing for the right options. Thus $x = \{0, r\omega^{y^L} | r\omega^{y^R}\}$, and x is commensurate with $\omega^y, y = \{y^L | y^R\}$. Our choice of y is unique since ω^y is commensurate with $\omega^{y'}$ if and only if $y = y'$. This is because we can add y' as a left option of y without changing its value. ■

Say that x is commensurate to ω^y. Let the set of all $r \in \mathbb{R}$ such that $r\omega^y < x$ be A, and $\mathbb{R} - A = B$. One of A, B has an extreme element. Call that element r. Then $x - r\omega^y$ is commensurate to $\omega^{y'}$ for some $y' < y$. If this stops, we get a nice, finite sum for x. If it doesn't stop, we want the infinite sum of all of these terms to be x. We will define infinite sums so that this works out.

The 0-term of x is the number $r_0\omega^{y_0}$ such that $x - r_0\omega^{y_0}$ is infinitesimal compared to x. It is allowed for $r_0 = 0$, which clearly only happens when $x = 0$. We define $\sum_{\beta<1} r_{y_\beta}\omega^{y_\beta} = r_0\omega^{y_0}$.

Now we define the α-term of x for all ordinals α by induction. Assume the β-term has been defined for all $\beta < \alpha$ as some $r_{y_\beta}\omega^{y_\beta}$. Then we define

$$\sum_{\beta<\alpha} r_{y_\beta}\omega^{y_\beta} \tag{1.4}$$

to be the simplest number whose β-term is $r_{y_\beta}\omega^{y_\beta}$ for all $\beta < \alpha$. Then we write

$$x = \sum_{\beta<\alpha} r_{y_\beta}\omega^{y_\beta} + x_\alpha, \tag{1.5}$$

and define the α-term of x to be the 0-term of x_α. If the α-term is 0, we can just stop the sum here, and we have gotten the Conway normal form of x.

Now consider the set of all partial sums

$$\sum_{\beta<\alpha} r_{y_\beta}\omega^{y_\beta} \tag{1.6}$$

for a particular number x. Assume that the sum never terminates at any ordinal, so that each α gives a different partial sum. Since the sums are chosen to be the simplest, they must be born before or at the same time as x. But the ordinals make up a proper class, while the numbers born by a certain day are a set. So the sum must stop a a certain point, and we have proven that each number has a Conway normal form.

2. Omnific Integers and Gaps

Now that we have an analogue to decimals, we can use that to define a surreal analogue to integers. A real number is an integer if its decimal expansion ends before its decimal point. So we could define a surreal number x to be an *omnific integer* if all the nonzero exponents of ω in Conway Normal Form are nonnegative. But this would make any real number an omnific integer, which we don't want. So we also add the restriction that y_0 must be an integer. This would work as a definition, but there is a much nicer way to describe omnific integers.

Definition 2.1. A surreal number x is an omnific integer if and only if $x = \{x-1|x+1\}$. We call the set of omnific integers $\mathbf{Oz}$.

This is how omnific integers are usually defined. We will prove this is equivalent to the other definition. First, we prove this definition makes sense.

Theorem 2.2. *The omnific integers form a group under addition. That is,*
i) If $x, y \in \mathbf{Oz}$, then $x + y \in \mathbf{Oz}$.
ii) If $x \in \mathbf{Oz}$, then $-x \in \mathbf{Oz}$.

Proof.
i) $x + y = \{x-1|x+1\} + \{y-1|y+1\} = \{x+y-1|x+y+1\}$, so $x + y \in \mathbf{Oz}$.
ii) $-x = \{-(x+1)| - (x-1)\} = \{-x-1| - x+1\}$, so $-x \in \mathbf{Oz}$. ∎

After this lemma, we will be ready to prove the equivalency.

Lemma 2.3. *We have the following:*
i) $r\omega^x \in \mathbf{Oz}$ if $x > 0$
ii) $r\omega^x \notin \mathbf{Oz}$ if $x < 0$.
iii) $r\omega^0 \in \mathbf{Oz}$ if and only if $r \in \mathbb{N}$.

Proof. i) The 0-term in $r\omega^x - 1$ and $r\omega^x + 1$ is $r\omega^x$. Since the equivalence classes we made are convex, all numbers in the interval $[r\omega^x - 1, r\omega^x + 1]$ are commensurate with ω^x. In fact, these numbers also have 0-term $r\omega^x$. Since $r\omega^x$ is the simplest number with that 0 term, we get $\{r\omega^x - 1|r\omega^x + 1\} = r\omega^x$ by the simplicity theorem.

ii) If $x < 0$, $r\omega^x - 1 < 0, r\omega^x + 1 > 0$, so $\{r\omega^x - 1|r\omega^x + 1\} = 0$.

iii) The interval $(r-1, r+1)$ contains exactly one integer, which is the value of $\{r-1|r+1\}$ by the simplicity theorem. ∎

Theorem 2.4. *Any surreal number x is a quotient of two omnific integers.*

So the surreal numbers are more like rationals than reals in this way. Maybe we can fix this by filling in the gaps between surreal numbers dedekind-cut style, defining gaps $\{X|Y\}$, where $X \cup Y = \mathfrak{On}$. The problem here is that the surreal numbers are too big to be a set, so X and Y aren't sets, but rather *proper classes*. This means that thing get extremely weird. For example, proper classes cannot be contained in other proper classes. To avoid all this weirdness, we will stay in the surreal numbers. If you want to learn more about classes, read (insert reference here).

3. The Fundamental Theorem Of Algebra

We will end this paper by proving an analogue of the fundamental theorem of algebra for the surreal numbers. If you haven't seen the proof of the fundamental theorem for the complex numbers, check out (insert reference). The proof of the fundamental theorem for an arbitrary field only requires two things:

1) Every positive number has a square root.

2) Every polynomial with odd degree has a root (in the field)

Unfortunately, **NO** isn't a field, since the surreal numbers are too big to be a set, and fields must be over a set. But this doesn't really matter, since we can define fields to be over a class instead without creating any problems. Fields with properties 1) and 2) are known as real closed fields. These fields turn out to be very nice. More specifically

We already know 1) is true. The proof of 2) requires a version of Hensel's lemma:

Lemma 3.1. *Let $f(x) = a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$ be a surreal polynomial. Assume that $a_i = b_i+\varepsilon_i$, where b_i is a real number and ε_i is infinitesimal. If $f'(x) = b_0x^n+b_1x^{n-1}+\cdots+b_n$ factors into $g_0(x)h_0(x)$, then $f(x) = h'(x)g'(x)$ for some g', h' with the same degree as g_0, h_0.*

Proof. First, we can assume $a_0 = 1$, and also that g_0, h_0 are relatively prime. We write f as a sum of polynomials where the coefficients that are commensurate with each other are all grouped together into a single polynomial. More formally, write

$$f(x) = \sum_{\beta<\alpha} \omega^{y_\beta} s_\beta(x) \tag{3.1}$$

where the y_β form a decreasing sequence, and $s_\beta(x)$ is a real polynomial with degree $\leq n$. We know that $y_\beta = 0$ and $s_0(x) = f'(x) = g_0(x)h_0(x)$. Say that $\deg g = r$ and $\deg h = s$, $r + s = n$.

We will inductively define a sequence g_α, h_α. Say that

$$\left(\sum_{\beta<\alpha} \omega^{y_\beta} g_\beta(x)\right)\left(\sum_{\beta<\alpha} \omega^{y_\beta} h_\beta(x)\right) \tag{3.2}$$

has the same coefficient of ω^y as f as long as $y \geq y_\beta$ for some $\beta < \alpha$, but is not equal to $f(x)$. Recall that the coefficient of ω^y will be a real polynomial. Say by induction that g_β, h_β have degrees at most $r - 1, s - 1$ for $\beta \neq 0$.

Let y_α be the biggest exponent i whose coefficient does not match that of f. Notice that $y_\alpha < y_\beta$ for all $\beta < \alpha$. We will find $f_\alpha(x), g_\alpha(x)$ with degrees less than r, s such that

$$\left(\sum_{\beta<\alpha} \omega^{y_\beta} g_\beta(x) + \omega^\alpha g_\alpha(x)\right)\left(\sum_{\beta<\alpha} \omega^{y_\beta} h_\beta(x) + \omega^\alpha h_\alpha(x)\right) \tag{3.3}$$

has the same coefficient of ω^y as f as long as $y \geq y_\alpha$. For this to be true, we just need that $g_\alpha(x)h_0(x) + h_\alpha(x)g_0(x) = p(x)$ for some polynomial p with degree at most $n - 1$. This is because the only way we can get a factor of x^n is in the product is through g_0f_0. Because g_0, h_0 are relatively prime, we can choose h_α, g_α to have degrees at most $r - 1, s - 1$. Notice that b_β is a finite sum of some y_β's, since we can't have an infinite decreasing chain o ordinals.

Now we claim that eventually

$$f(x) = \left(\sum_{\beta<\alpha} \omega^{y_\beta} g_\beta(x)\right)\left(\sum_{\beta<\alpha} \omega^{y_\beta} h_\beta(x)\right), \tag{3.4}$$

so that the process stops eventually. This must happen because the ordinals form a class, while the finite sums of y_β's are a set, and we cannot repeat y_β's. ■

Now we are ready.

Theorem 3.2. *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with odd (integer) degree and $a_i \in \mathbf{On}$. Then $f(x)$ has a root in $\mathbf{On}$.*

Proof. This is equivalent to showing that the only irreducible odd degree surreal polynomials are those with degree 1. First, we can scale x such that all the coefficients are ≤ 1. Any number $a \leq 1$ can be written as $b+\varepsilon$, where b is real and ε is infinitesimal. (Take the conway normal forms.) So we can assume that $f(x) = x^n + (b_{n-1} + \varepsilon_{n-1})x^{n-1} + \cdots + (b_0 + \varepsilon_0)$. We can also shift x such that $b_{n-1} + \varepsilon_{n-1}$ is 0. Then we can shift x again so that at least one coefficient other than that of x^n has a nonzero real part, unless $f = x^n$, which is an obvious case. By the previous lemma, the real part of this must not have relatively prime factors. So the polynomial is either of form $(x + a)^n$, n odd or $(x^2 + bx + c)^n$. The last option is impossible, having even degree. So the real part of f is $(x + a)^n$. Since the coefficient of x^{n-1} is 0, we get that $a = 0$. But this is in contradiction to the assumption that some other coefficient had a nonzero real part. ■

Thus we have shown that $\mathbf{No}$ is real closed, so the *surcomplex numbers*, defined as $x + iy, x, y \in \mathbf{On}, i^2 = -1$, are an algebraically closed field. In fact, since the surreal numbers contain all ordered fields, the surcomplex numbers turn out to contain all algebraically closed fields, although this is beyond the scope of our paper.

Theorem 3.3. *The surcomplex numbers are algebraically closed.*

4. Conclusion

There are still a lot of things we don't know about the surreal numbers. A lot of people have been trying to do calculus on the surreal numbers. The big problem is that the usual ε-δ definition of limits doesn't work anymore, since we have infinitesimals. Omnific number theory also has some interesting problems, but it seems that proofs of theorems are either the same is in the case of regular integers, or unknown, so not much work has been done there. Similarly, we know basically nothing about the surcomplex numbers, since we don't even know that much about the surreals.