ON COMBINATORIAL GAME THEORY AND CHESS

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ABSTRACT. While Combinatorial Game Theory (CGT) is often difficult to apply to standard chess games, where reduced-material endgames are reached, one can apply the principles of CGT to effectively analyze variations and evaluate positions. In this paper, we demonstrate the principles of CGT using chess positions, and use CGT to analyze endgames. We also cover an open problem posed by Elkies on the subject of dyadic rationals in chess.

1. Introduction

The principles of Combinatorial Game Theory (CGT) do not align very well with chess, one of the oldest games. One reasons for this is that the vast majority of chess positions cannot be treated as the sum of individual, smaller games; pieces like rooks, queens, and bishops have too strong of an influence all across the board, and can swing from one side of the board to the other in a move. Another reason is that CGT is most effectively applied to cold games like Blue-Red Hackenbush, where having an extra move gives at most an infinitesimal advantage, and usually a disadvantage. Chess, on the other hand, is a "hot" game: positions where one side would rather skip a move, known as **zugzwangs**, are unusual, and positions where both sides would rather skip a move, known as **trébuchet**, are very rare.

Nevertheless, when reduced material is on the board, or when major pieces are unusually restricted, this gives rise to positions that can analyzed effectively using CGT principles. Most commonly we'll be looking at King and Pawn endgames, building up from basic positions to analyze larger ones with multiple components.

We assume that the reader is aware of the rules of Chess, knows the chess coordinate system and chess notation, and is somewhat familiar with the game. We also assume that, from each chess position, both players play the best moves.

Additionally, since we are considering Chess from a CGT standpoint of view, the words *White* and *Left* are synonymous and may be interchanged, as are the words *Black* and *Right*. Most of this paper will be examples - there are no real "theorems" involving chess, rather demonstrations of CGT principles through chess positions.

2. Concepts

Just as one does when learning CGT through the context of a game for the first time, it makes sense to look at examples that demonstrate the most basic concepts in chess.

2.1. Numbers.

The easiest way to think about numbers in the context of chess is through option notation. Without positions like 0 and 1, it's impossible to build up the numbers, both in a Number-Theoretic lens and a CGT lens. Practically any combinatorial game has some concept of 0, Chess being no exception.

Example. Notice that $0 = \{|\}$. This corresponds to the following:

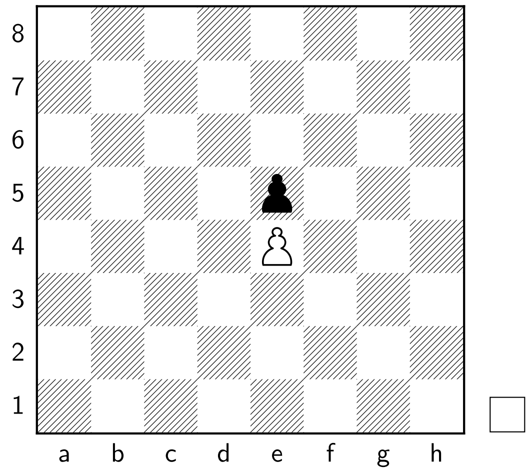

Figure 1. Neither side has any move, so this position is equal to the 0 game.

A position equivalent to 1 is not hard to find either:

Example. In the figure below, Left can move to the position 0, while Right has no moves. So, this game is equal to $1 = \{0\,|\}$.

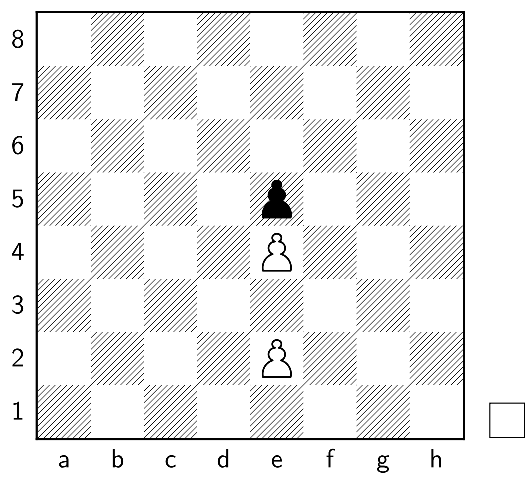

Figure 2. A game equivalent to 1.

In these simplistic positions, we can think of basic numbers as "free moves" (more commonly known in chess as *tempi*). In Figure 2, white has one extra move, so the position is equivalent to 1. We can take this to the extreme as well:

Example. White has 4 free moves he can make, assuming that the pawn on e2 does not move two squares forward. (In fact, this gives us a view of dominated options as well - it does benefit white to move his pawn 2 squares instead of 1.)

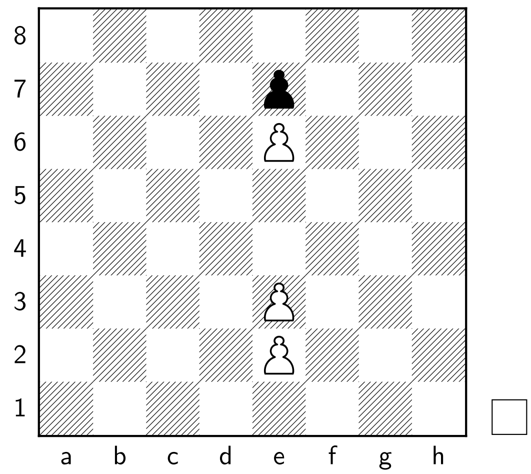

Figure 3. A game equal to 4, the maximal value on a single file.

2.2. **Disjunctive Sum.** We have numbers - the next thing we'd like to do is work with them somehow. The disjunctive sum is quite important for this - just like in normal mathematics, if we can't relate numbers with an operation, we're helpless.

In normal chess gameplay, the disjunctive sum often fails to be of use because components of a game are not independent - for example, a rook could swing over from the a-file to the h-file and easily take part in the battle on both sides of the board. In endgames with reduced material, however, we get an opportunity to see the disjunctive sum on the chessboard.

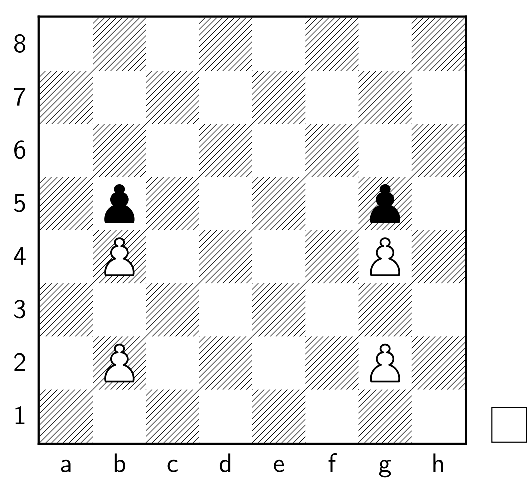

Figure 4. $1 + 1 = 2$.

Example. In Figure 4, we see two instances our position in Figure 2 on two files - the b and g files. Note these two files are "independent" of each other, so this position is $1+1=2$. In option notation, we can also confirm that this game is $\{1\,|\} = 2$.

With just this much knowledge, we can begin to analyze positions arising from real games. The following is an example from Elkies [1]:

Example. The main battle being waged is on the kingside - whoever wins the opponent's g-pawn will win the game by marching their g-pawn up the board. Note that we have a trébuchet between the kings, so the evaluation of the position comes down to the free moves on the files. Note that black has 2 free moves on the a-file, while white has 4 and 1 free moves on the c and e files, respectively. So, the position evaluates to $-2+4+1=3$. Hence, no matter who is to move, white wins by at least 2 tempi - in particular, he wins be at least 4 if it is black to move, and 2 if it is white to move.

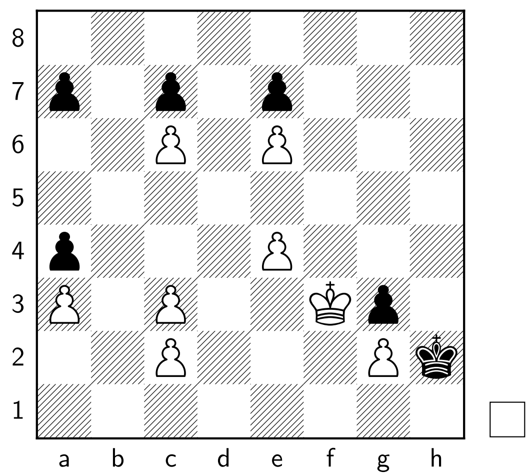

Figure 5. $-2+4+1=3$.

The canonical 8×8 board is quite restrictive, and larger boards generally tend to introduce many more possibilities, including larger numbers. Elkies [2] dives into this topic in more detail - for now, we'll be considering 8×8 boards only.

Other types of games that come to mind, like ordinals, surreal numbers, and omnific integers, don't have clear cut interpretations in chess. However, we can find other constructs of CGT as well, starting with:

2.3. **Infinitesimals.** Infinitesimals relate to positions where moving has marginal benefit; as mentioned earlier, chess is a "hot" game, where having an extra move usually grants a sizable advantage to a player, but there are positions where this isn't the case.

The following two examples give a good demonstration of two basic infinitesimals in the context of chess:

Example. Analyzing the options of the position below yields that this position is $\{0 \mid 0\} = *$.

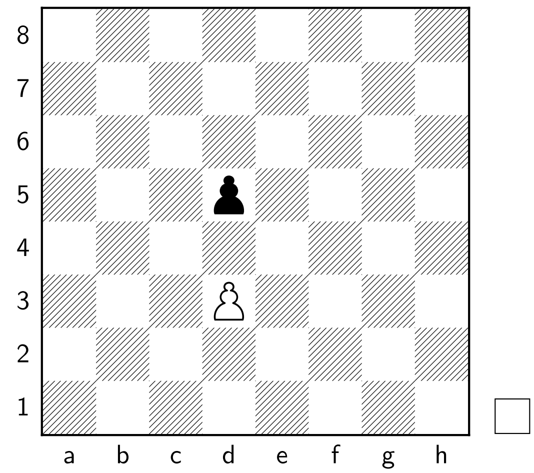

Figure 6. A $*$ position.

Taking advantage of option notation, we can reach another infinitesimal position:

Example. In this position, black has the choice of moving to $*$, while white has two choices: moving to 0, or moving to $*$. Moving to $*$ doesn't give white an advantage, so this position is $\{0, *\} = \uparrow$ (To reach $\downarrow$, we can simply reflect this position, so that the black pawn is on the 7th rank and the white pawn is on the 4th.

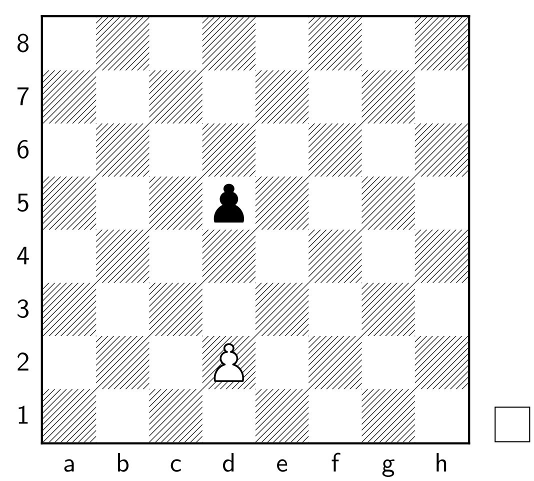

Figure 7. A $\uparrow$ position.

2.4. **Switches.** Our final construct, switches, are unique - in these scenarios, both sides gain the same advantage (or disadvantage) from having the move.

Example. Note that, in the below position, whoever is to move gains an extra "tempo move" once they advance their pawn. Hence, this is just the position $\{1 \mid -1\} = \pm 1$.

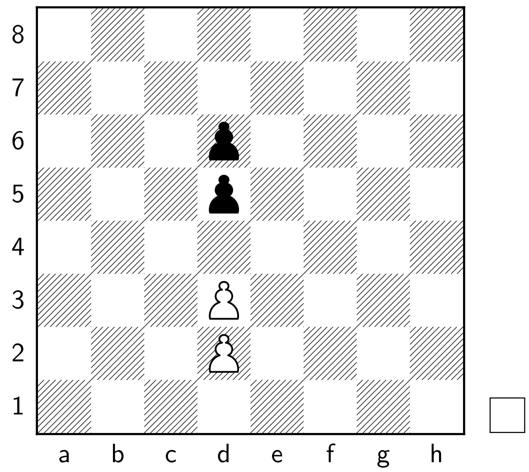

Figure 8. An example of a switch.

3. An example from a game

We now follow an example of Elkies [1], which will tie in the concepts laid out.

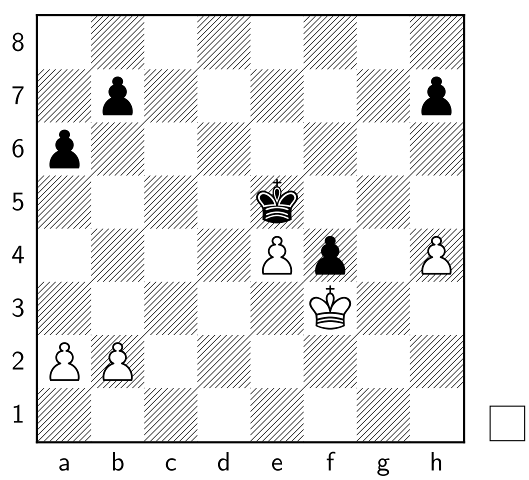

Figure 9. Schweda-Sika, Brno 1929.

There are three independent components to this game:

- The queenside component, defined by the a- and b-files;
- The central component, defined by the e- and f-files;
- The kingside component, defined by the h-file.

We tackle each one separately. Starting with the kingside, in option notation, note that White can only move to $\downarrow$, while black has two choices: to move to 0 by advancing two squares, or to move to $*$ by advancing one square. We can simply take 0 as an option and ignore $*$, so the component evaluates as $\{\downarrow \mid 0, *\} = \{\downarrow \mid 0\} = \Downarrow *$.

The central component is a trebuchet position - whoever moves their king first loses their central pawn (and with it the game), so this evaluates trivially to 0.

The queenside component is the most complicated, but by removing dominated options, one can arrive at it being equal to $\uparrow$. This scenario is achieved by 1.a4 b5 2.axb5 axb5, which results in the following position:

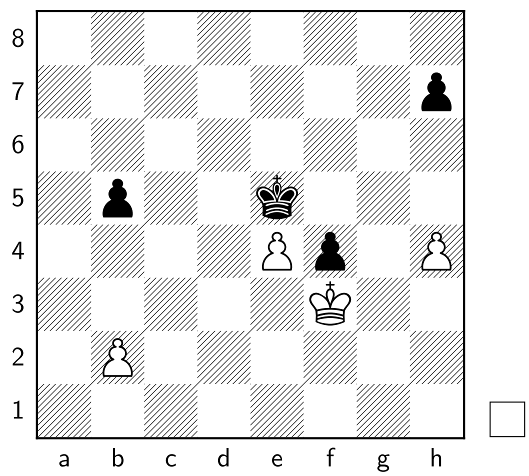

Figure 10. The queenside is clearly a $\uparrow$ position.

What does all of this mean? Note that $\uparrow + \Downarrow * = \downarrow *$. This game is confused with 0 (i.e. fuzzy), so it follows that this position is a first-player win position.

This is one of only a handful of positions where analyzing from a CGT perspective is more fruitful than examining each branch of a tree of variations. The above position is one of them: by determining the values of the kingside and queenside with CGT, it becomes much easier to determine the outcome of the game, regardless of who's move it is.

4. Open Problem

There are quite a few interesting open problems relating to chess from a CGT perspective, which can be found in Elkies' paper [1]. One of the more approachable ones to a chess beginner is the following:

Question 4.1. *The topic of dyadic rationals often comes up combinatorial games. For chess, it's not hard to construct dyadic rationals for sufficiently large chessboard sizes. However, with the standard limitations of* 8×8*, it's not necessarily true that dyadic rationals can be represented. Is there a position on the* 8×8 *chessboard corresponding to* $\frac{1}{8}$*?*

With lengthy yet straightforward analysis, Elkies gives a hysterical position that evaluates to $-\frac{1}{4}$, as shown in Figure 11.

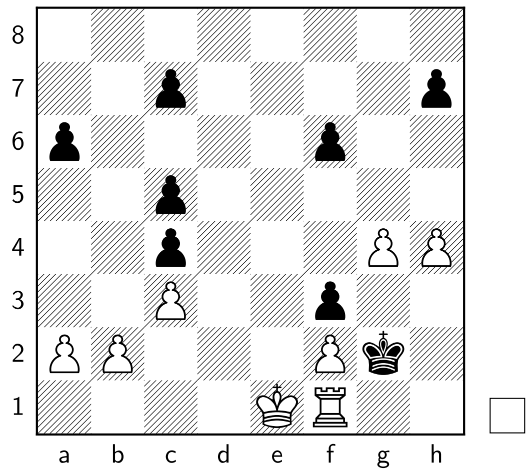

Figure 11. A ludicrous yet instructive example.

To look at dyadic rationals, we can consider a more simplified version of the position - namely the queenside, excluding the black pawn on c7.

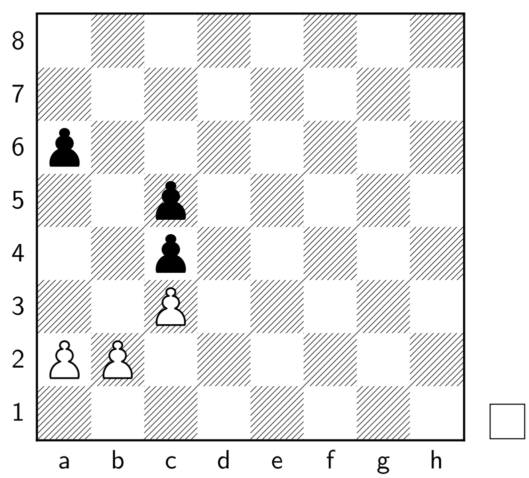

Figure 12. What does this position evaluate to?

We'll analyze white's options first.

(1) If white plays 1. a4, we arrive at a $*$ position (as the only pieces that matter are the pawns on the a-file).
(2) Similarly, if white plays 1. a3, we arrive at a 0 position.
(3) If white plays 1. b3, black can respond 1..cxb3 2. axb3 c4, from which 3. b4 is forced, again leading to a 0 position.

Now we analyze black's options. Black is forced to play 1..a5. Looking at White's options:

(1) White can play 2. a3, leading to a $*$ position.
(2) White can play 2. a4, leading to a 0 position
(3) The move 2. b3 is a mistake, due to 2...cxb3 3. axb3 c4 4. bxc4 a4 and black promotes.

Black, meanwhile, can only play a4, leading to a position with value 1. So, the right option is $\{0, * \mid 1\} = \frac{1}{2}$. In total, the position in Figure 12 evaluates to $\{0, * \mid 1/2\} = 1/4$.

The astute reader may notice that, on larger boards, smaller dyadic rationals can be reached. For example, looking at Figure 12, suppose we moved black's a-pawn one square backward, but didn't give it the option of moving two squares forward from it's original position.

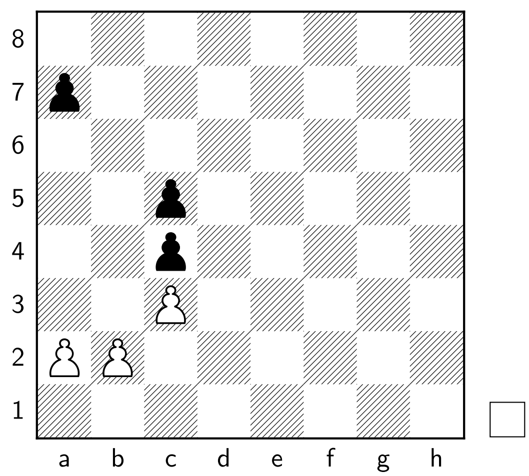

Figure 13. A bit of bending the rules.

What happens here? White's options essentially remain the same; in the end, they evaluate to 0. However, black has the option only of moving to the position Figure 12, which is just 1/4. This position then just evaluates to $\{0 \mid 1/4\} = 1/8$.

Of course, we did some cheating here by removing one of the most important rules of the game - the ability for a pawn to move two squares from it's starting position. But if we add a 9th rank and a i-th file to this board, suddenly it's no longer cheating - black's pawn isn't on the starting square, and we do get 1/8 as a value. In this way, we can see how to construct arbitrarily small dyadic rationals $1/2^N$ on boards of large enough dimension.

Subject to the normal 8×8 restrictions, it's still unclear how to reach a position with value 1/8, and the problem remains unsolved. Yet many corresponding positions of larger dyadic rationals exist, which may gave light on how to achieve 1/8 on the board.

References

[1] Noam D. Elkies. On numbers and endgames: Combinatorial game theory in chess endgames, 1999.
[2] Noam D. Elkies. Higher nimbers in pawn endgames on large chessboards, 2000.