On various impartial games

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Abstract

Impartial games are a very important set of games in Combinatorial Game Theory[1]. An example of an impartial game is Nim, which is one of the most important games, in Combinatorial Game Theory. It is well known that the second player will win if and only if $a_1 \oplus a_2 \oplus \ldots \oplus a_n = 0$. In this paper we explore various impartial games such as Fibonacci Nim, Wythoff's Game.

1 Introduction

Impartial games are a very important set of games in Combinatorial Game Theory[1]. An example of an impartial game is Nim, which is one of the most important games, in Combinatorial Game Theory. The rules of Nim are as follows:

Consider n piles of stones. The first pile has a_1 stones, the second has a_2 stones, and so on until the n-th pile has a_n stones. The current player chooses any pile and removes any positive number of stones from that pile (but the player cannot take more than the current number of stones in that pile). The person who makes the last move wins.

It is well known that the second player will win if and only if

$$a_1 \oplus a_2 \oplus \ldots \oplus a_n = 0$$

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In this paper we explore winning and losing positions in games such as Fibonacci Nim and Wythoff's Game.

2 Fibonacci Nim

Fibonacci Nim is a famous variants of the traditional Nim game. The rules of Fibonacci Nim is as follows[1−7]:

Assume there are n coins in a pile. On the first move, the first player can take no more than $n-1$ coins. For every next move, the player takes at most twice as many coins as the number of coins taken on the previous move. In other words, let q_m be the maximum number of coins that can be removed on the m-th move and let r_m be the number of coins removed on the m-th move. Then, $q_1 = n - 1$ and $q_i = 2r_{i-1}$ where $i \geq 2$. The person who takes the last coin wins the game.

We use the following definition for a position in Fibonacci Nim.

Definition 2.1. *If at the current state, there are n coins remaining in the pile and the maximum number of coins we can currently remove is r, then we represent the current position as (n, r).*

From here, we see the starting position is just $(n, n-1)$. We claim the following theorem.

Theorem 2.2. *A fibonacci nim game is in $\mathcal{P}$ if and only if n, the number of coins initially, is a Fibonacci number.*

We first define the Zeckendorf representation of a number. This is a unique representation for the values of N written as a sum of fibonacci numbers, such that no to fibonacci numbers have adjacent indices.

Definition 2.3. *Define $z(k)$ to be the smallest Fibonacci number in the Zeckendorf representation of k.*

With the following definition we can state a more general theorem from which Theorem 2.2 follows since the first player loses if and only if $n - 1 < z(n) \leq n$, so we must have $z(n) = n$ which means n needs to be a Fibonacci number.

Theorem 2.4. *We have (n, q_m) is a losing position for the current player if and only if $q_m < z(n)$.*

To prove Theorem 2.4, we first prove the following lemma.

Lemma 2.5. *Suppose $n > 1$ and $1 \leq k < z(n)$. Suppose that $z(k) = F_t$. Then $z(n-k) \in \{F_{t-1}, F_{t+1}\}$. In particular $z(n-k) \leq 2k$.*

Proof. We induct on the number of Zeckendorf parts of k. The base case is if k is a Fibonacci number, say $k = F_t$, and suppose that $z(n) = F_s$. We now consider the cases where $s \equiv t \pmod 2$ and $s \not\equiv t \pmod 2$.

Assume $s \equiv t \pmod 2$. Let $t = s - 2d$ for some positive integer d.

$$\begin{aligned} F_s - k &= F_s - F_t \\ &= F_s - F_{s-2d} \\ &= (F_s - F_{s-2}) + (F_{s-2} - F_{s-4}) + \ldots + (F_{s-2d+2} - F_{s-2d}) \\ &= F_{s-1} + F_{s-3} + \ldots + F_{s-2d+1} \end{aligned}$$

Thus, we have $z(F_s - k) = z(n-k) = F_{s-2d+1} = F_{t+1}$.

Now, assume $s \not\equiv t \pmod 2$. Let $t = s-2d-1$ for some non-negative integer d.

$$\begin{aligned} F_s - k &= F_s - F_t \\ &= F_s - F_{s-2d-1} \\ &= (F_s - F_{s-2}) + (F_{s-2} - F_{s-4}) + \ldots + (F_{s-2d+2} - F_{s-2d}) + (F_{s-2d} - F_{s-2d-1}) \\ &= F_{s-1} + F_{s-3} + + \ldots + F_{s-2d+1} + F_{s-2d-2} \end{aligned}$$

Thus, we have $z(F_s - k) = z(n-k) = F_{s-2d-2} = F_{t-1}$. This proves that the statement is true when k is a Fibonacci number.

Now assume the lemma holds when k has $m-1$ parts in its Zeckendorf representation. Consider a number l that has m parts in its Zeckendorf representation. Suppose $z(l) = F_t$. Thus, $z(l - z(l)) \geq F_{t+1}$ and $l - z(l)$ has $m-1$ parts, so by induction we have:

$$z(n - l + z(l)) \geq F_{t+1} = F_t = z(l)$$

. Thus, we get that $z(n-k) \in \{F_{t-1}, F_{t+1}\}$, as desired. □

Now, we prove theorem 2.4.

Proof. Let A be the set of positions (n, q_m) such that $q_m < z(n)$ and B be the set of positions (n, q_m) such that $q_m > z_n$. We prove that for all moves from any position in A we go to a position in B and for any position in B there exists a move to go to a position in A.

We begin by showing that if $(n, q_m) \in B$ then there is a move to a A position. Since $q_m > z(n)$, there is $(n - z(n), 2z(n))$. If $n = z(n)$, then this move is clearly to a A position, since the game is finished. Otherwise, if $z(n) = F_t$, then $z(n-z(n)) = y(n) \geq F_{t+2} > 2F_t$, so $(n-z(n), 2z(n)) \in A$, where $y(n)$ represents the second smallest Fibonacci number in the Zeckendorf representation of n.

Now suppose that $(n, q_m \in A$, so $q_m < z(n)$. Then, for any $k \leq q_m < z(n)$, we have $z(n-k) \leq 2k$ by the previous lemma, so $(n-k, 2k) \in B$, as desired. □

3 Wythoff's Game

Wythoff's Game is another famous variant of the Nim game. It is well known by many number theorists[8]. The rules of Wythoff's Game is as follows:

There are two piles of coins. The players play alternately and either takes some arbitrary amount from one of the piles or an equal amount from both piles. The person who takes the last coin wins the game.

The following theorem summarizes the results for Wythoff's Game.

Theorem 3.1. *If one pile has n coins and the other has m coins, then this is a winning position if and only if it cannot be written as the form $n = \lfloor k\phi \rfloor$ and $m = \lfloor k\phi^2 \rfloor$, for any non-negative integer k, where ϕ is the golden ratio.*

But, before we prove the theorem we first start with a lemma.

Lemma 3.2. *The losing positions in Wythoff's game are the pairs* (a_n, b_n) *for all* $n \geq 0$*, which are defined recursively*

$$a_n = mex\{a_i, b_i : 0 \leq i < n\}, \qquad b_n = a_n + n$$

Proof. Call the positions described in the lemma the A positions, and the others the B positions. We begin by showing that there is no move from a A position to another A position.

Suppose there was a move from (a_n, b_n) to some (a_m, b_m) or (b_m, a_m). Notice the situation when we start with (b_n, a_n) is symmetric.

Since the minimum of the coordinates cannot increase, we must have $m < n$. By the construction of a_n, we cannot have $a_n = a_m$ or $a_n = b_m$, so this move must remove an equal amount from both piles. But this is impossible as well, since $|a_n - b_n| = n$, but $|a_m - b_m| = m < n$. Thus all moves from A positions go to B positions.

We are now left to show that for any B position, there is a move to a A position. Suppose that (a, b) is not of the form (a_n, b_n) or (b_n, a_n) for any n. By symmetry, we must assume $a \leq b$. First, suppose that $a = a_n$ for some n. If $b > b_n$, then there is a move from (a, b) to $(a, b_n) = (a_n, b_n)$ which is a move to a A position. Otherwise, if $a \leq b < b_n$, then let $m = b - a$. Then $m < n$ so $a_m < a_n$, and there is a move to (a_m, b_m). Finally suppose that $a \neq a_n$ for all n. Then, by the construction of a_n, we must have $a = b_n$ for some n. Thus, $a_n < a = b_n$ so there is a move to (b_n, a_n) which completes the proof. □

We now prove Theorem 3.1.

Proof. Note that $\phi^2 = \phi + 1$, which implies that $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$. Thus, by Rayleigh's Theorem, the Beatty sequences for ϕ and ϕ^2 are complementary[9]. Note also that

$$\lfloor n\phi^2 \rfloor = \lfloor n\phi + n \rfloor = \lfloor n\phi \rfloor + n$$

Thus, we have

$$\lfloor n\phi \rfloor = \text{mex}\{\lfloor m\phi \rfloor, \lfloor m\phi^2 \rfloor : 0 \leq m < n\} \quad \text{and} \quad \lfloor n\phi^2 \rfloor = \lfloor n\phi \rfloor + n$$

Thus, the result directly follows from Lemma 3.2. □

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