Paper - The Angel and the Devil Problem

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Abstract

This research paper explores the Angel and Devil problem, a theoretical game on an infinite chessboard, originally proposed by John Conway, where an Angel attempts to evade a Devil who progressively blocks squares that the Angel is not on. This problem is significant within combinatorial game theory, as it illustrates challenges in pathfinding and survival strategies with an enemy (the Devil). Through a survey of existing strategies and solutions, this paper examines key approaches, focusing on how Angels of varying power can achieve escape despite the Devil's attempts to enclose the Angel. This work offers an overview of the problem's mathematical depth and its implications for infinite game spaces and combinatorial strategies.

1 Introduction

The *Angel and Devil problem*, first proposed by John Conway in 1982, is a classic problem in combinatorial game theory that explores a complex pursuit-evasion scenario on an infinite chessboard. In this game, two players—an Angel and a Devil—compete within a grid, where the Angel seeks to evade capture, and the Devil, as an adversary, aims to confine the Angel by blocking any square the Angel is *not* on each turn. The problem becomes particularly challenging given the infinite nature of the board and the movement rules, where the Angel can jump to any square within a certain distance, known as its "power," while the Devil can block a single square per turn. The central question is whether the Angel has a guaranteed escape strategy or if the Devil can eventually trap it.

The *Angel and Devil problem* has intrigued mathematicians for decades due to its insights into pathfinding, *spatial strategy*, and adversarial decision-making in infinite spaces. While seemingly simple, it presents deep challenges in combinatorial optimization, particularly regarding the balance between the Angel's movement freedom and the Devil's blocking power. Notably, early attempts to solve the problem yielded no clear solutions, and it was only in 2006 that Brian Bowditch [1] provided a solution that sufficiently 'powerful' Angels could evade the Devil indefinitely. More will be discussed on this further in the paper. Further contributions by mathematicians such as András Máthé [6] and Oddvar Kloster [4] expanded on the problem, leading to a rich body of work on strategies and movement constraints. In this paper, we examine the foundational aspects of the Angel and Devil problem, using existing research to gain a deeper understanding of known solutions. Additionally, we will be proving these solutions, and understanding why the Angel can only win if its power is greater than or equal to 2.

2 Mathematical Framework

In the introduction, we had given a basic, non-mathematical framework for the Angel and Devil problem. In this section, we will represent the Angel and Devil problem with a mathematical foundation to analyze the problem more rigorously.

The game is played on an infinite two-dimensional Cartesian grid, represented as $\mathbb{Z}^2$. Each square on the grid corresponds to a lattice point (x, y), where $x, y \in \mathbb{Z}^2$. At the start of the game, all squares are deemed unblocked. However, as the game progresses, the Devil permanently blocks certain squares.

2.1 Angel's Movement

The Angel occupies a square on the grid at any given turn. Its movement is defined as follows. The Angel starts at an initial position, say, $A_0 = (x_0, y_0)$, and at each turn t, the Angel moves to a new position, $A_t = (x_t, y_t)$ such that:

$$||A_t - A_{t-1}||_1 \leq k,$$

where $k \geq 1$ is the Angels *power*. and $|| \cdot ||$ is the *Manhattan distance*.

$$||A_t - A_{t-1}||_1 = |x_t - x_{t-1}| + |y_t - y_{t-1}.$$

Additionally, the Angel cannot move to a blocked square, that is, $A_t \notin \mathcal{B}$, where $\mathcal{B}$ is the set of all blocked squares.

Here are some basic interpretations of small k values.

1. $k = 1$ represents a king's role in chess, in which the Angel can move up to one square in distance, either diagonally, horizontally, or vertically.
2. $k = 2$ allows the Angel to move up to 2 spaces per move
3. We can define $k = n$ as the Angel's ability to move n spaces per move.

$k = 2$ allows the Angel to move up to

2.2 Devil's Movement

The Devil blocks squares on the grid to trap the Angel. Its actions can be represented as this; At each turn, the Devil selects one square $D_t = (x_d, y_d)$.

Once blocked, a square remains permanently inaccessible to the Angel. Formally, this is $\mathcal{B}_{t+1} = \mathcal{B} \cup D_t$, where $\mathcal{B}_0 = \emptyset$.

The Devil, similar to the Angel, also has certain constraints. The Devil can only block one square per turn, unlike the Angel, who's capability for distance traveled is dependent on its power.

Additionally, the Devil cannot block a square the Angel is on, which implies that $D_t \neq A_t$.

2.3 Win Positions and Conditions

2.3.1 The Angel's Winning Strategy

The Angel's primary objective is to escape the Devil indefinitely, ensuring that at every turn, it has at least one valid move. Mathematically, this can be expressed as maintaining access to an *infinite connected component* in the Reachability Graph $G = (V, E)$, where

- V: All unblocked squares.
- E: Valid moves between squares determined by the Angel's power k, defined by the Manhattan distance constraint:

$$\|A_t - A_{t+1}\|_1 \leq k.$$

Using combinatorial game theory, we classify the Angel's position at any turn into one of the *outcome classes*:

- **P-class (previous-player win):** The Devil has already reduced the reachable space so that the Angel can no longer win.
- **N-class (next-player win):** The Angel can always find a move that forces the game toward its survival.

The Angel's winning strategy involves selecting moves that preserve the N-class status of the position. For this, the Angel must avoid moves leading to disconnected or isolated regions of the grid and Prioritize paths that maximize the number of reachable squares, similar to *dominance* (in CGT), where moves with greater long-term value are preferred.

2.3.2 The Devil's Winning Strategy

The Devil wins if it can trap the Angel in a finite region by progressively blocking squares until all potential moves are invalid. This is similar with transitioning the game into a P-class position. The Devil's strategy uses principles from combinatorial game theory, such as:

- **Domination of Options:** Block squares that limit the Angel's most valuable moves first, akin to reducing a complex game to its simplest equivalent form.

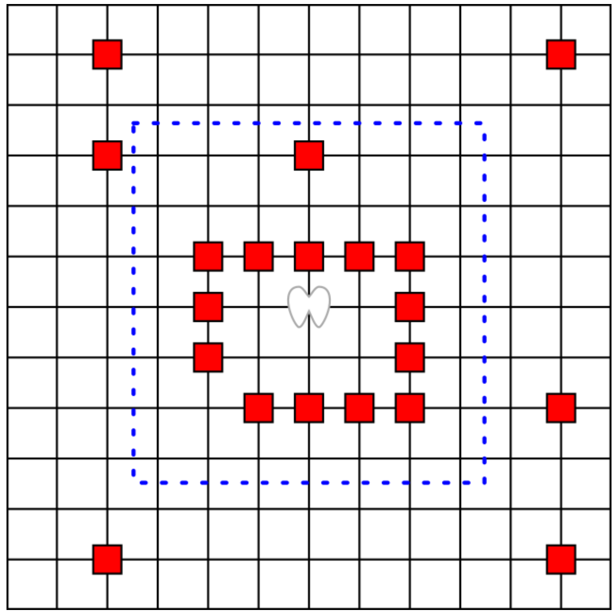

Figure 1: Shows how the Angel is trapped within progressively smaller and smaller regions

- **Partitioning the Reachability Graph (defined above):** Divide the grid into finite components, progressively isolating the Angel in smaller and smaller regions. A good example of this possibility is Figure 1 [2], in which the blue-dotted boundary line is the finite component that the Devil would want to surround to minimize the chances of the Angel escaping.

The Devil can apply the concept of *remoteness* from CGT, which measures how close the game is to termination. Blocking moves closer to the Angel increases remoteness, accelerating the win condition.

2.3.3 Game Ordinances

To formalize the game further, we use options and values. A position P can be represented as:

$$P = \{P_L \mid P_R\},$$

where P_L are the positions reachable by the Angel's moves and P_R are those influenced by the Devil's blocks. For the Angel/Devil problem:

- The Angel's reachable positions correspond to *left options*, increasing its maneuverability.
- The Devil's blocks reduce *right options*, limiting the Angel's future choices.

3 Escape Strategies for the Angel

To understand the Angel's ability to evade the Devil indefinitely, we must develop escape strategies that makes sure that an infinite connection of unblocked squares remains accessible to the Angel at all times. These strategies depend on the Angel's power k. It has been proven that for $k \geq 2$, the Angel has a winning strategy, which we aim to demonstrate rigorously.

3.1 Reachability and Infinite Connectivity

Let $G_t = (V_t, E_t)$ represent the *Reachability Graph* at turn t, where:

- V_t: The set of all unblocked squares at turn t.
- E_t: Edges between vertices $u, v \in V_t$ exist if $||u - v||_1 \leq k$, corresponding to the Angel's movement constraints.

To win, the Angel must ensure G_t contains an infinite connected component at every turn t. Specifically, the Angel must avoid entering regions where the Devil can create finite disconnected components of blocked squares. Additionally, the Angel must use paths that maximize the size of the reachable region in G_t.

3.2 The Power of the Angel: $k \geq 2$

For $k \geq 2$, the Angel's mobility allows it to avoid entrapment by using two key properties:

- Expansion Rate: The number of squares reachable by the Angel in a single turn grows quadratically with k. Specifically, the reachable region is a diamond-shaped area of size [5]:

$$S_k = \{(x, y) \in \mathbb{Z}^2 : |x| + |y| \leq k\},$$

  containing $2k^2 + 2k + 1$ squares.

- Path Flexibility: By increasing the Angel's power, we can allow it to escape narrowing corridors of unblocked squares that the Devil may attempt to create.

3.2.1 A Strategy for $k = 2$

For $k = 2$, the Angel can use a *zigzag escape strategy* [3]. In this strategy, the Angel chooses moves that avoid creating a narrow band of unblocked squares behind it. Hence, by alternating between directions (e.g., moving two squares horizontally, then two vertically), the Angel ensures that its future reachable region remains large and connected, as wanted.

Formally, the Angel chooses its next position A_{t+1} such that the size of the connected component containing A_{t+1} in G_{t+1} is maximized.

3.2.2 Generalization to $k > 2$

For $k > 2$, the Angel's increased movement range provides even greater flexibility, allowing it to:

- Escape regions where the Devil attempts to enclose it.
- Choose paths that maximize long-term accessibility to unblocked squares.

The Angel can adopt a spiral-like escape pattern, continuously moving outward to ensure it never enters a finite region. Such a strategy is always feasible due to the Angel's ability to "jump" over narrow bands of blocked squares created by the Devil.

3.3 Proof of Winning Strategy for $k \geq 2$

To prove the Angel's winning strategy for $k \geq 2$, we rely on two key lemmas:

Lemma: Infinite Reachability If the Angel has power $k \geq 2$, it can always move to a square within an infinite connected component of unblocked squares at every turn.

Proof. The Devil can block only one square per turn, while the Angel has access to $2k^2 + 2k + 1$ squares. For $k \geq 2$, this ensures the rate of blocking by the Devil is insufficient to isolate the Angel in a finite region. The Angel can always move toward the boundary of the reachable region, preserving connectivity. □

Lemma: Escape from Finite Regions The Angel with $k \geq 2$ can always escape from any finite region surrounded by blocked squares.

Proof. For $k \geq 2$, the Angel's movement range allows it to jump over narrow bands of blocked squares, regardless of their configuration. The Devil cannot block squares quickly enough to create a barrier wider than k, ensuring the Angel's escape. □

Combining these lemmas, we conclude that the Angel has a winning strategy for $k \geq 2$.

3.4 The Case of $k = 1$: Why the Angel Cannot Win

For $k = 1$, the Angel's movement is restricted to a neighborhood of size 9, consisting of its current square and the 8 adjacent squares, similar to a king's moves in chess. This limited movement range gives the Devil a significant advantage in blocking the Angel's escape routes. It has been rigorously proven by Máthé [6] that the Devil can always trap the Angel when $k = 1$.

3.4.1 The Devil's Winning Strategy for $k = 1$

The Devil can use a *confinement strategy* to trap the Angel in a finite region by systematically blocking squares. This strategy has been described in works such as Bowditch [1] and Kloster [4]. It involves:

1. Blocking squares adjacent to the Angel to reduce its movement options.
2. Progressively creating a closed "ring" of blocked squares around the Angel.
3. Ensuring that the Angel's reachable space is finite at all times.

3.4.2 Formal Proof

To prove that the Devil can always trap the Angel when $k = 1$, we proceed as follows:

Lemma: Finite Confinement for $k = 1$ For $k = 1$, the Angel will eventually be confined to a finite region of the grid by the Devil.

Proof. The Devil can block one square per turn. For $k = 1$, the Angel's movement is limited to the immediate neighborhood of its current position, a region of at most 8 unblocked squares. Over successive turns:

- The Devil blocks a square adjacent to the Angel, reducing the number of accessible squares.
- Each time the Angel moves, it can only escape to one of the remaining adjacent unblocked squares, which is also finite.

The confinement strategy described in Bowditch [1] ensures that the Devil can always select a blocking move that reduces the Angel's available space. Eventually, the Angel will reach a position where all 8 surrounding squares are blocked, leaving it with no valid moves. At this point, the Devil wins. □

Lemma: No Infinite Reachable Component for $k = 1$] For $k = 1$, the Angel cannot maintain access to an infinite connected component of unblocked squares.

Proof. The Devil's blocking strategy ensures that the reachable component around the Angel is progressively reduced. Because the Angel can only move to adjacent squares, the Devil's single blocked square per turn is sufficient to break connectivity or isolate the Angel in a finite region. Over time, the Devil partitions the grid into smaller and smaller finite components, eventually enclosing the Angel in one of these finite regions [6]. □

Combining these lemmas, we see that the Devil's confinement strategy is guaranteed to succeed, and the Angel cannot evade the Devil indefinitely when $k = 1$.

3.4.3 Illustrative Example

Consider a scenario where the Angel starts at the origin $(0, 0)$. At each turn:

- The Devil blocks one of the 8 adjacent squares.
- The Angel moves to an adjacent unblocked square, such as $(1, 0)$.

By carefully choosing the square to block, the Devil systematically reduces the Angel's movement options. After approximately 8 moves (depending on the initial configuration), the Angel is completely surrounded by blocked squares, as shown in Figure 2.

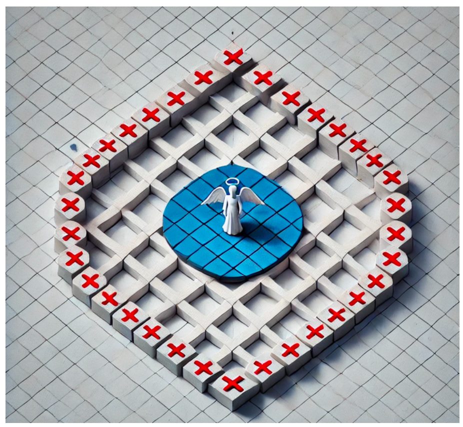

Figure 2: Example of the Devil trapping the Angel for $k = 1$. The Devil surrounds the Angel (blue circle) by blocking adjacent squares (red X's), eventually leaving no valid moves.

3.4.4 Why $k = 1$ Is Fundamentally Limited

The limitation for $k = 1$ lies in the Angel's inability to move more than one square per turn. This makes it impossible to escape a finite region once the Devil begins enclosing it. Unlike cases where $k \geq 2$, the Angel cannot "jump over" narrow barriers of blocked squares, which ultimately leads to its capture.

The Devil's strategy shows that the Angel cannot escape indefinitely when $k = 1$. This result is different from the $k \geq 2$ case, showing how even a small increase in power changes the Angel's ability to maintain access to an infinite connected component of unblocked squares.

References

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